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Matricial Nehari Problems, J-Inner Matrix Functions and the Muckenhoupt Condition

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The classes of regular and strongly regular γ-generating matrices and J-inner matrix valued functions arose in the investigation of the matricial Nehari problem, bitangential interpolation problems, and inverse problems for canonical systems as well as the theory of characteristic functions of operators and operator nodes. In this paper, new characterizations of these classes are developed. In particular, the property of strong regularity is characterized in terms of a matricial Muckenhoupt (A_2) condition in the Treil–Volberg form. These results are based on parametrizations that are intimately connected with Darlington representations of matrix valued functions in the Schur and Carathéodory classes. As a byproduct of this analysis, examples of strongly regular γ-generating matrices and entire J-inner matrix valued functions that are unbounded on the circle and the real line, respectively, are presented. © 2001 Academic Press

1. INTRODUCTION

The classes $\mathfrak{M}_R(p, q)$ and $\mathfrak{M}_{sR}(p, q)$ of regular and strongly regular γ-generating mvf's (matrix valued functions) arise naturally in the study of the matricial Nehari problem for contractive $p \times q$ mvf's on the circle $\mathbb{T}$ and on the line $\mathbb{R}$. They intervene in the representation of the set of all solutions

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to completely indeterminate and strictly completely indeterminate Nehari problems, respectively (as the matrices of the coefficients of the linear fractional transformations that are used in these representations); see e.g., Theorems 3.6 and 3.7 below. The class $\mathfrak{M}_R(p, q)$ was extensively investigated in [Ar6]. Here, we continue this investigation, obtain some new results on this class and initiate the investigation of the class $\mathfrak{M}_{sR}(p, q)$. In particular, we characterize the class $\mathfrak{M}_{sR}(p, q)$ in terms of a matricial (A_2) Muckenhoupt condition in the Treil-Volberg [TrV] form. The results on these two classes lead naturally to a corresponding set of results for the classes $\mathcal{U}_{rR}(J)$ and $\mathcal{U}_{sR}(J)$ of right regular and strongly regular J-inner mvf's. The class $\mathcal{U}_{rR}(J)$ plays a significant role in the theory of completely indeterminate bitangential interpolation problems. It was introduced in [Ar4, 5] and extensively studied in [Ar6–8] and a number of subsequent publications; [ArD1] is a convenient reference for the main facts and additional developments. The class $\mathcal{U}_{sR}(J)$ plays an analogous role in the theory of strictly completely indeterminate bitangential interpolation problems; see e.g., Subsection 2.2 below. This class was introduced and characterized in two different ways in [ArD1]. It plays a significant role in our study of inverse problems for canonical systems, [ArD1], ..., [ArD4]. The class of operator nodes with characteristic functions in the class $\mathcal{U}_{sR}(J)$ that belong to the Hardy class of index 2, was characterized by Z. Arova [Ara2].

In this paper we continue the study of the classes $\mathcal{U}_{rR}(J)$ and $\mathcal{U}_{sR}(J)$. In particular, we establish a new characterization of the class $\mathcal{U}_{sR}(J)$ in terms of the matricial Muckenhoupt condition that is based on the characterization of the class $\mathfrak{M}_{sR}(p, q)$ that was mentioned earlier. In order to obtain these results, we consider a number of parametrizations of the mvf's $\mathfrak{A} \in \mathfrak{M}(p, q)$, the class of γ-generating mvf's, and $U \in \mathcal{U}(J)$, the class of J-inner mvf's, for the following choices of the signature matrix J: j_{pq}, $j_p = j_{pp}$, J_p and $\mathcal{J}_p$; see (2.1) and (7.1). These parametrizations are intimately connected with the Darlington representation of mvf's of the Schur class $\mathcal{S}^{p\times q}$ and the Carathéodory class $\mathcal{C}^{p\times p}$. As a byproduct of our characterization of the classes $\mathfrak{M}_{sR}(p, q)$ and $\mathcal{U}_{sR}(j_{pq})$, we exhibit a one-parameter family of strongly regular γ-generating 2×2 mvf's that are unbounded on the circle and a one-parameter family of entire strongly regular J-inner mvf's that are unbounded on the line.

The paper is organized as follows: The end of this section is devoted to notation.

Section 2 is devoted to a review of the necessary facts that are needed from the theory of J-inner functions and the more general class of J-contractive functions with respect to the region Ω_+, where Ω_+ denotes either the open unit disc

$$\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

TABLE 1.1

Ω_+	$\mathbb{D}$	$\mathbb{C}_+$
Ω_0	$\mathbb{T}$	$\mathbb{R}$
$\rho_\omega(\lambda)$	$1-\lambda\bar{\omega}$	$-2\pi i(\lambda-\bar{\omega})$
$\lambda^\sim$	$1/\bar{\lambda}$ (if $\lambda\neq 0$)	$\bar{\lambda}$
$f^{\#}(\lambda)$	$f(\lambda^\sim)^*$ (if $\lambda\neq 0$)	$f(\lambda^\sim)^*$

or the open upper half plane

$$\mathbb{C}_+=\{\lambda\in\mathbb{C}:\Im\lambda>0\}.$$

Section 3 reviews a number of needed facts (some known and some new) from the theory of γ-generating mvf's and the Nehari problem based on data that is given on the boundary Ω_0 of Ω_+.

In Section 4, we give the first characterization of the classes $\mathfrak{M}_{sR}(p,q)$ and $\mathcal{U}_{sR}(j_{pq})$ in terms of the matricial Muckenhoupt condition. These characterizations are based on the connection between the Treil–Volberg result and the matricial Nehari problem. The parametrizations of the mvf's in the classes $\mathfrak{M}(p,p)$ and $\mathcal{U}(j_p)$ that are considered in Sections 5 and 6, respectively, are then used to obtain a number of other characterizations of the classes $\mathfrak{M}_R(p,p)$ (Theorem 5.5), $\mathfrak{M}_{sR}(p,p)$ (Theorem 5.8), $\mathcal{U}_{rR}(j_p)$ (Theorem 6.4) and $\mathcal{U}_{sR}(j_p)$ (Theorem 6.6).

In Section 7, the preceding results for the mvf's in the class $\mathcal{U}(j_p)$ are reformulated for the classes $\mathcal{U}(J_p)$ and $\mathcal{U}(\mathcal{J}_p)$. In this section we also consider a dual set of parametrizations of mvf's from $\mathcal{U}(J)$ that is appropriate for the characterization of left regularity and left strong regularity. Formulas for the parameters corresponding to the three signature matrices of interest are summarized in Tables 7.1–7.3; the last table corresponds to the dual parametrization. In Subsection 7.3, we specialize some of our results to the class $\mathcal{E}\cap\mathcal{U}(J)$ of entire J-inner mvf's. The examples mentioned earlier are presented in Subsections 5.6 and 7.6.

We remark that in Sections 5–7 we focus our attention on the case $q=p$. However, a number of these results can be used to obtain corresponding conclusions for the case $q\neq p$ by invoking the embeddings that were used for that purpose in Section 4.

1.1. *Notation.* The kernel $\rho_\omega(\lambda)$, the reflection $\lambda^\sim$ of λ with respect to the boundary Ω_0 of Ω_+ and the notation $f^{\#}(\lambda)$ are defined in Table 1.1.

Notice that

$$\Omega_+=\{\omega\in\mathbb{C}:\rho_\omega(\omega)>0\}\qquad\text{and}\qquad\Omega_0=\{\omega\in\mathbb{C}:\rho_\omega(\omega)=0\}.$$

We shall set

$$\Omega_- = \{\omega = \mathbb{C} : \rho_\omega(\omega) < 0\},$$

i.e., Ω_- is the exterior of Ω_+.

The following classes of mvf's will be used:

$$L_r^{p\times q}(\Omega_0) = \{p\times q \text{ mvf's } f\colon f \text{ is measurable on } \Omega_0 \text{ and } \|f\|_r < \infty\}$$

for $r = 1, 2, \infty$, where

$$\|f\|_r = \begin{cases} \left\{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} [\operatorname{trace}\{f(e^{i\theta})^* f(e^{i\theta})\}]^{r/2}\, d\theta\right\}^{1/r} & \text{if } \Omega_0 = \mathbb{T} \\ \left\{\displaystyle\int_{-\infty}^{\infty} [\operatorname{trace}\{f(\mu)^* f(\mu)\}]^{r/2}\, d\mu\right\}^{1/r} & \text{if } \Omega_0 = \mathbb{R} \end{cases}$$

for $r = 1, 2$ and

$$\|f\|_\infty = \operatorname{ess\,sup}\{\|f(\mu)\| \colon \mu \in \Omega_0\}.$$

$L_2^{p\times q}(\Omega_0)$ is a Hilbert space with respect to the inner product defined by the norm $\|f\|_2$.

$$\widetilde{L_r^{p\times q}}(\Omega_0) = \begin{cases} L_r^{p\times q}(\Omega_0) & \text{if } \Omega_0 = \mathbb{T} \\ L_r^{p\times q}\left(\Omega_0, \dfrac{d\mu}{\pi(1+\mu^2)}\right) & \text{if } \Omega_0 \in \mathbb{R}, \end{cases}$$

for $r = 1, 2$.

$$H_r^{p\times q}(\Omega_+) = \{p\times q \text{ mvf's } f\colon f \text{ is holomorphic in } \Omega_+ \text{ and } \|f\|_{H_r} < \infty\}$$

for $r = 2$ and ∞, where

$$\|f\|_{H_2}^2 = \begin{cases} \displaystyle\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \operatorname{trace}\{f(re^{i\theta})^* f(re^{i\theta})\}\, d\theta & \text{if } \Omega_+ = \mathbb{D} \\ \displaystyle\sup_{v>0} \int_{-\infty}^{\infty} \operatorname{trace}\{f(\mu+iv)^* f(\mu+iv)\}\, d\mu & \text{if } \Omega_+ = \mathbb{C}_+ \end{cases}$$

and

$$\|f\|_{H_\infty} = \sup\{\|f(\lambda)\| \colon \lambda \in \Omega_+\}.$$

$$K_2^{p\times q}(\mathbb{D}) = \{p\times q \text{ mvf's } f\colon f^\# \in \lambda H_2^{q\times p}(\mathbb{D})\}.$$

$$K_2^{p\times q}(\mathbb{C}_+) = \{p\times q \text{ mvf's } f\colon f^\# \in H_2^{q\times p}(\mathbb{C}_+)\}.$$

$H_2^{p\times q}(\Omega_+)$ and $K_2^{p\times q}(\Omega_+)$ can be identified with the closed subspaces $H_2^{p\times q}(\Omega_0)$ and $K_2^{p\times q}(\Omega_0)$ of $L_2^{p\times q}(\Omega_0)$, respectively, by identifying f with its nontangential boundary limits. Then

$$L_2^{p\times q}(\Omega_0)=H_2^{p\times q}(\Omega_0)\oplus K_2^{p\times q}(\Omega_0)$$

and

$$\|f\|_{H_2}=\|f\|_2$$

for $f\in H_2^{p\times q}(\Omega_+)$. The space $H_\infty^{p\times q}(\Omega_+)$ can also be identified with a subspace $H_\infty^{p\times q}(\Omega_0)$ of $L_\infty^{p\times q}(\Omega_0)$ and

$$\|f\|_{H_\infty}=\|f\|_\infty$$

for $f\in H_\infty^{p\times q}(\Omega_+)$.

$$\mathcal{S}^{p\times q}(\Omega_+)=\{f\in H_\infty^{p\times q}(\Omega_+):\|f\|_\infty\leqslant 1\},$$

$$\mathcal{S}_{in}^{p\times q}(\Omega_+)=\{f\in\mathcal{S}^{p\times p}(\Omega_+):f(\mu)^*\,f(\mu)=I_p\text{ a.e. on }\Omega_0\},$$

$$\mathcal{S}_{out}^{p\times p}(\Omega_+)=\{f\in\mathcal{S}^{p\times p}(\Omega_+):\overline{fH_2^{p\times 1}}(\Omega_0)=H_2^{p\times 1}(\Omega_0)\},$$

where $\bar{\mathcal{X}}$ denotes the closure of the indicated set $\mathcal{X}$ in $H_2^{p\times 1}(\Omega_0)$.

$$\mathcal{C}^{p\times p}(\Omega_+)=\{p\times p\text{ mvf's }f\text{: }f\text{ is holomorphic in }\Omega_+$$
$$\text{and }f(\lambda)+f(\lambda)^*\geqslant 0\text{ for every }\lambda\in\Omega_+\}.$$

$$\mathcal{N}^{p\times q}(\Omega_+)=\{g/h\text{: }g\in\mathcal{S}^{p\times q}(\Omega_+)\text{ and }h\in\mathcal{S}^{1\times 1}(\Omega_+)\}.$$

$$\mathcal{N}_+^{p\times q}(\Omega_+)=\{g/h\text{: }g\in\mathcal{S}^{p\times q}(\Omega_+)\text{ and }h\in\mathcal{S}_{out}^{1\times 1}(\Omega_+)\}.$$

$$\mathcal{N}_{out}^{p\times q}(\Omega_+)=\{g/h\text{: }g\in\mathcal{S}_{out}^{p\times q}(\Omega_+)\text{ and }h\in\mathcal{S}_{out}^{1\times 1}(\Omega_+)\}.$$

The classes $\mathcal{S}^{p\times q}(\Omega_-)$, $\mathcal{N}^{p\times q}(\Omega_-)$ and $\mathcal{C}^{p\times p}(\Omega_-)$ are defined in an analogous way.

$$\Pi^{p\times q}=\{p\times q\text{ mvf's }f:f\text{ is meromorphic in }\Omega_+\cup\Omega_-,$$
$$f(\lambda)=f_\pm(\lambda)\text{ in }\Omega_\pm,\ f_\pm\in\mathcal{N}^{p\times q}(\Omega_\pm)\text{ and}$$
$$f_+(\mu)=f_-(\mu)\text{ for a.e. }\mu\in\Omega_0\}.$$

If f is defined in a region Ω, then

$$\mathfrak{H}_f=\{\lambda\in\Omega:f\text{ is holomorphic at }\lambda\}.$$

If F is a matrix or mvf, then

$$\Re F = \frac{F+F^*}{2} \qquad \text{and} \qquad \Im F = \frac{F-F^*}{2i}$$

denote the real and imaginary parts of F, respectively.

$$\mathscr{E}^{p\times q} = \{ p\times q \text{ mvf's that are entire, i.e., holomorphic in all of } \mathbb{C}\}.$$

From now on, we adopt the convention that for a class $\mathscr{X}^{p\times q}$ of $p\times q$ mvf's, $\mathscr{X}^p$ is short for $\mathscr{X}^{p\times 1}$ and $\mathscr{X}$ is short for $\mathscr{X}^{1\times 1}$. We shall also write $\mathscr{E}\cap\mathscr{X}^{p\times q}$ and $\Pi\cap\mathscr{X}^{p\times q}$ instead of $\mathscr{E}^{p\times q}\cap\mathscr{X}^{p\times q}$ and $\Pi^{p\times q}\cap\mathscr{X}^{p\times q}$, respectively.

2. J-INNER MATRIX VALUED FUNCTIONS

2.1. *J-inner mvf's and associated reproducing kernel Hilbert spaces.* Let J be an $m\times m$ signature matrix, i.e., J is both selfadjoint and unitary with respect to the standard inner product in the space $\mathbb{C}^m$. We shall assume that $J\neq \pm I_m$. Then J is unitarily equivalent to the diagonal signature matrix

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \tag{2.1}$$

where

$$p+q=m, \quad p=\operatorname{rank}(I_m+J) \quad \text{and} \quad q=\operatorname{rank}(I_m-J), \tag{2.2}$$

i.e., there exists an $m\times m$ unitary matrix V such that

$$V^*JV = j_{pq}. \tag{2.3}$$

An $m\times m$ mvf (matrix valued function) $U(\lambda)$ is said to be J-contractive in the domain Ω_+ if it is meromorphic in Ω_+ and

$$U(\lambda)^*\, JU(\lambda) \leqslant J \tag{2.4}$$

for every point $\lambda\in\Omega_+\cap\mathfrak{H}_U$. It is well known that this last condition is equivalent to the fact that the kernel

$$K_\omega(\lambda) = \frac{J-U(\lambda)\, JU(\omega)^*}{\rho_\omega(\lambda)} \tag{2.5}$$

is positive semidefinite on $\Omega_+ \times \Omega_+$, or, to be more precise, on $(\Omega_+ \cap \mathfrak{H}_U) \times (\Omega_+ \cap \mathfrak{H}_U)$.

The class of J-contractive mvf's was extensively investigated by V. P. Potapov [Po]. Accordingly we shall designate this class of mvf's by the symbol $\mathscr{P}(J)$. It is known that if $U \in \mathscr{P}(J)$, then $U(\lambda)$ has nontangential limits (i.e., boundary values) at a.e. point in Ω_0; see e.g., [Ar2], [Dy1].

A J-contractive mvf $U \in \mathscr{P}(J)$ is said to be J-inner if its boundary values are J-unitary a.e. on Ω_0, i.e., if

$$U(\mu)^* JU(\mu) = J \tag{2.6}$$

for a.e. $\mu \in \Omega_0$. The class of J-inner mvf's will be denoted by the symbol $\mathscr{U}(J)$. Every $U \in \mathscr{U}(J)$ admits a pseudocontinuation into Ω_- that is defined by the rule

$$U(\lambda) = J\{U^{\#}(\lambda)\}^{-1} J \tag{2.7}$$

for every point $\lambda \in \Omega_-$ for which the indicated inverse exists. The extension to Ω_- belongs to $\mathscr{N}^{m \times m}(\Omega_-)$ and (the full) $U \in \Pi^{m \times m}$.

From now on we shall assume that J-inner mvf $U(\lambda)$ is defined on $\mathfrak{H}_U$ in all of $\mathbb{C}$ and shall let $\mathscr{H}(U)$ denote the reproducing kernel Hilbert space with reproducing kernel given by formula (2.5). This means that the lineal of $m \times 1$ vector valued functions (vvf's) of the form

$$f(\lambda) = \sum_{j=1}^{n} K_{\omega_j}(\lambda)\, \xi_j, \qquad \omega_j \in \mathfrak{H}_U \quad \text{and} \quad \xi_j \in \mathbb{C}^m,$$

is dense in $\mathscr{H}(U)$ and

$$\langle f, K_\omega \xi \rangle_{\mathscr{H}(U)} = \xi^* f(\omega)$$

for $\omega \in \mathfrak{H}_U$.

The vectors $f \in \mathscr{H}(U)$ are meromorphic $m \times 1$ vvf's (vector valued functions) of bounded Nevanlinna characteristic on Ω_+ and Ω_- such that $f(\mu)$ has the same nontangential boundary values at a.e. point $\mu \in \Omega_0$ as $\lambda \to \mu$ from Ω_+ and from Ω_-, i.e., $\mathscr{H}(U) \subset \Pi^m$.

The reproducing kernel Hilbert space $\mathscr{H}(U)$ for the case $\Omega_+ = \mathbb{C}_+$ was introduced and extensively investigated by L. de Branges [dB1], [dB2]. The analogous formulation for the case $\Omega_+ = \mathbb{D}$, including an important technical improvement by Rovnyak [Rov], was considered by Ball [Ba]. For a unified treatment of both, see [AlD1] and, for additional information, [AlD3], [Dy1] and [Ara1].

2.2. *Strongly regular J-inner mvf's.* Let

$$W(\lambda)=\begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix} \tag{2.8}$$

be the block decomposition of a mvf $W \in \mathcal{U}(j_{pq})$ with diagonal blocks $w_{11}(\lambda)$ of size $p \times p$ and $w_{22}(\lambda)$ of size $q \times q$. It is well known that the linear fractional transformation

$$T_W[\mathscr{E}]=(w_{11}\mathscr{E}+w_{12})(w_{21}\mathscr{E}+w_{22})^{-1} \tag{2.9}$$

maps $\mathscr{E} \in \mathscr{S}^{p\times q}(\Omega_+)$ into $\mathscr{S}^{p\times q}(\Omega_+)$, i.e.,

$$T_W[\mathscr{S}^{p\times q}(\Omega_+)] \subset \mathscr{S}^{p\times q}(\Omega_+), \tag{2.10}$$

where

$$T_W[\mathscr{S}^{p\times q}(\Omega_+)]=\{T_W[\mathscr{E}]\colon \mathscr{E} \in \mathscr{S}^{p\times q}(\Omega_+)\}. \tag{2.11}$$

A mvf $W \in \mathcal{U}(j_{pq})$ is said to be strongly regular if there exists a least one mvf

$$s \in T_W[\mathscr{S}^{p\times q}(\Omega_+)] \qquad \text{such that} \quad \|s\|_\infty < 1. \tag{2.12}$$

The symbol $\mathcal{U}_{sR}(j_{pq})$ will designate the class of strongly regular j_{pq} inner mvf's.

This definition of strong regularity arose in connection with the following generalized Nevanlinna–Pick interpolation problem in the Schur class that is formulated in terms of a given pair of inner mvf's $b_1 \in \mathscr{S}_{in}^{p\times p}(\Omega_+)$, $b_2 \in \mathscr{S}_{in}^{q\times q}(\Omega_+)$ and a mvf $s^\circ \in \mathscr{S}^{p\times q}(\Omega_+)$: Describe the set

$$\mathscr{S}(b_1,b_2;s^\circ)=\{s \in \mathscr{S}^{p\times q}(\Omega_+) : b_1^{-1}(s-s^\circ)\,b_2^{-1} \in H_\infty^{p\times q}(\Omega_+)\}. \tag{2.13}$$

This interpolation problem will be referred to as the GSIP $(b_1, b_2; s^\circ)$. It is said to be strictly completely indeterminate if there exists a mvf

$$s \in \mathscr{S}(b_1,b_2;s^\circ) \qquad \text{such that} \quad \|s\|_\infty < 1. \tag{2.14}$$

There is an intimate connection between strictly completely indeterminate generalized Nevanlinna–Pick interpolation problems in the Schur class and the class $\mathcal{U}_{sR}(j_{pq})$ that is based on the fact that the diagonal blocks of $W \in \mathcal{U}(j_{pq})$ satisfy

$$(w_{11}^{\#})^{-1} \in \mathscr{S}^{p\times p}(\Omega_+) \qquad \text{and} \qquad (w_{22})^{-1} \in \mathscr{S}^{q\times q}(\Omega_+).$$

More precisely, let

$$(w_{11}^{\#})^{-1} = b_1 \varphi_1, \qquad \text{where} \quad b_1 \in \mathcal{S}_{in}^{p\times p} \quad \text{and} \quad \varphi_1 \in \mathcal{S}_{out}^{p\times p}$$

$$(w_{22})^{-1} = \varphi_2 b_2, \qquad \text{where} \quad b_2 \in \mathcal{S}_{in}^{q\times q} \quad \text{and} \quad \varphi_2 \in \mathcal{S}_{out}^{q\times q}.$$

Then the pair $\{b_1, b_2\}$ is uniquely determined by W up to a right [resp. left] unitary constant multiplier for $b_1(\lambda)$ [resp. $b_2(\lambda)$]. We shall refer to $\{b_1, b_2\}$ as an associated pair for W and shall write $\{b_1, b_2\} \in ap(W)$.

THEOREM 2.1. *Let the GSIP $(b_1, b_2; s^\circ)$ be strictly completely indeterminate. Then there exists an mvf $W \in \mathcal{U}(j_{pq})$ such that*

$$\begin{aligned} &1. \quad \mathcal{S}(b_1, b_2; s^\circ) = T_W[\mathcal{S}^{p\times q}]. \\ &2. \quad \{b_1, b_2\} \in ap(W). \end{aligned} \tag{2.15}$$

This mvf $W(\lambda)$ is unique up to a right constant j_{pq} unitary factor and is automatically strongly regular (i.e., $W \in \mathcal{U}_{sR}(j_{pq})$).

THEOREM 2.2. *Let $W \in \mathcal{U}_{sR}(j_{pq})$, let $\{b_1, b_2\} \in ap(W)$ and let $s^\circ \in T_W[\mathcal{S}^{p\times q}]$. Then the GSIP $(b_1, b_2; s^\circ)$ is strictly completely indeterminate and formula (2.15) holds for the considered mvf's W, b_1, b_2 and s°.*

To this point we have only defined strong regularity for $\mathcal{U}(j_{pq})$. But if $U \in \mathcal{U}(J)$ and (2.3) is in force, then the mvf

$$W(\lambda) = V^* U(\lambda)\, V \tag{2.16}$$

is j_{pq}-inner. We shall say that $U(\lambda)$ is a strongly regular J-inner mvf and shall write $U \in \mathcal{U}_{sR}(J)$ if and only if $W \in \mathcal{U}_{sR}(j_{pq})$.

The following theorem was established in Section 6 of [ArD1]:

THEOREM 2.3. *Let $U \in \mathcal{U}(J)$. Then $U \in \mathcal{U}_{sR}(J)$ if and only if the boundary values $f(\mu)$ of every $f \in \mathcal{H}(U)$ belong to the space $L_2^m(\Omega_0)$, i.e., if and only if*

$$\mathcal{H}(U) \subset L_2^m(\Omega_0). \tag{2.17}$$

Moreover, if $U \in \mathcal{U}_{sR}(J)$, then there exists a pair of positive constants γ_1 and γ_2 such that

$$\gamma_1 \|f\|_{L_2^m} \leqslant \|f\|_{\mathcal{H}(U)} \leqslant \gamma_2 \|f\|_{L_2^m} \tag{2.18}$$

for every $f \in \mathcal{H}(U)$.

Remark 2.4. The following characterizations of the class $\mathcal{U}_{rR}(J)$ of right regular J-inner mvf's and $U_S(J)$ of singular J-inner mvf's are also established in Section 6 of [ArD1]:

$$U \in \mathcal{U}_{rR}(J) \Leftrightarrow \mathcal{H}(U) \cap L_2^m(\Omega_0) \text{ is dense in } \mathcal{H}(U). \tag{2.19}$$

$$U \in \mathcal{U}_S(J) \Leftrightarrow \mathcal{H}(U) \cap L_2^m(\Omega_0) = \{0\}. \tag{2.20}$$

To be more precise, [ArD1] treats the case $\Omega_+ = \mathbb{C}_+$, but the same proofs work for $\Omega_+ = \mathbb{D}$.

Remark 2.5. The following inclusions are established in Subsection 3.8 of [ArD1]:

$$L_\infty^{m \times m}(\Omega_0) \cap \mathcal{U}(J) \subset \mathcal{U}_{sR}(J) \subset \widetilde{L_2^{m \times m}}(\Omega_0). \tag{2.21}$$

3. THE NEHARI PROBLEM

The Nehari problem NP (f°, Ω_0) based on a fixed mvf $f^\circ \in L_\infty^{p \times q}(\Omega_0)$ with $\|f^\circ\|_\infty \leqslant 1$ is to describe the set

$$\mathcal{F}(f^\circ) = \{f \in L_\infty^{p \times q}\colon f - f^\circ \in H_\infty^{p \times q}(\Omega_+) \text{ and } \|f\|_\infty \leqslant 1\}. \tag{3.1}$$

The Hankel operator Γ_f with symbol $f \in L_\infty^{p \times q}(\Omega_0)$ is defined by the rule

$$\Gamma_f = P_{K_2^p} M_f|_{H_2^q}, \tag{3.2}$$

where M_f denotes the operator that acts from $L_2^q(\Omega_0)$ into $L_2^p(\Omega_0)$ by multiplication by the mvf f and $P_{K_2^p}$ denotes the orthogonal projection of $L_2^p(\Omega_0)$ onto $K_2^p(\Omega_0)$.

It is known that if f and $f^\circ \in L_\infty^{p \times q}(\Omega_0)$, then:

1. $$f - f^\circ \in H_\infty^{p \times q}(\Omega_0) \Leftrightarrow \Gamma_f = \Gamma_{f^\circ}. \tag{3.3}$$

2. $$\|\Gamma_{f^\circ}\| = \min\{\|f\|_\infty\colon f \in \mathcal{F}(f^\circ)\}. \tag{3.4}$$

The Nehari problem NP (f°, Ω_0) is said to be determinate if

$$\mathcal{F}(f^\circ) = \{f^\circ\}$$

and indeterminate otherwise. A finer subdivision of the indeterminate Nehari problems is needed: We shall say that the NP (f°, Ω_0) is completely indeterminate if

$$\{(f - f^\circ)\,\eta\colon f \in \mathcal{F}(f^\circ)\} \neq \{0\}$$

for every vector $\eta \in \mathbb{C}^q$ of length one and that it is strictly completely indeterminate if there exists an $f \in \mathscr{F}(f^\circ)$ with $\|f\|_\infty < 1$.

In view of formula (3.4), $\|\Gamma_{f^\circ}\| \leqslant 1$ and the following statement is now selfevident:

LEMMA 3.1. *The NP* (f°, Ω_0) *is strictly completely indeterminate if and only if* $\|\Gamma_{f^\circ}\| < 1$.

The next lemma permits one to reformulate all known results for the Nehari problem on the circle $\mathbb{T}$ to the Nehari problem on the line $\mathbb{R}$. The proof is selfevident and it is omitted.

LEMMA 3.2. *The NP* $(f^\circ, \mathbb{R})$ *is equivalent to the NP* $(f^\circ(\psi(\cdot)), \mathbb{T})$, *where*

$$\psi(\zeta) = \frac{i(1-\zeta)}{1+\zeta}, \qquad \zeta \in \mathbb{T}, \tag{3.5}$$

that is to say

$$\{f - f^\circ\} \in H_\infty^{p\times q}(\mathbb{C}_+) \qquad \textit{and} \qquad \|f\|_\infty \leqslant 1$$

if and only if

$$\{f(\psi(\cdot)) - f^\circ(\psi(\cdot))\} \in H_\infty^{p\times q}(\mathbb{D}) \qquad \textit{and} \qquad \|f \circ \psi\|_\infty \leqslant 1.$$

In the next subsection we shall use Lemma 3.2 to reformulate a number of known results for the Nehari problem on $\mathbb{T}$ for Ω_0.

3.1. *Some preliminaries for the Nehari problem.* Let $f \in L_\infty^{p\times q}(\mathbb{T})$ and let

$$c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, e^{-ik\theta}\, d\theta.$$

Now, for a fixed $f^\circ \in L_\infty^{p\times q}(\mathbb{T})$, let

$$\gamma_k = c_{-k}(f^\circ) \qquad \text{for} \quad k = 1, 2, \dots .$$

Then clearly

$$\mathscr{F}(f^\circ) = \mathscr{G}(\gamma_1, \gamma_2, \dots),$$

where

$$\mathscr{G}(\gamma_1, \gamma_2, \dots) = \{f \in L_\infty^{p\times q}(\mathbb{T}) : c_{-k}(f) = \gamma_k \text{ for } k = 1, 2, \dots \\ \text{and } \|f\|_\infty \leqslant 1\}.$$

The original Nehari problem was to describe the set $(\mathscr{G}(\gamma_1, \gamma_2, ...)$ for an arbitrary sequence of $p\times q$ matrices $\gamma_1, \gamma_2, ...$. It is well known (see [AAK2] and Page [Pa]) that this set is nonempty if and only if the Hankel operator

$$\Gamma: \eta = \mathrm{col}[\eta_1, \eta_2, ...] \in \ell_2^q \to \xi = \mathrm{col}[\xi_1, \xi_2, ...] \in \ell_2^p$$

that is defined by the rule

$$\xi_j = \sum_{k=1}^{\infty} \gamma_{j+k-1}\eta_k$$

is contractive, i.e., if and only if $\|\Gamma\| \leqslant 1$. This result was obtained by Nehari [Ne] for the case $p=q=1$. If $f^\circ \in \mathscr{G}(\gamma_1, \gamma_2, ...)$, then

$$\mathscr{G}(\gamma_1, \gamma_2, ...) = \mathscr{F}(f^\circ).$$

Moreover, if

$$\eta_+(\zeta) = \sum_{k=1}^{\infty} \eta_k \zeta^{k-1} \qquad \text{for} \quad \eta \in \ell_2^q$$

and

$$\xi_-(\zeta) = \sum_{k=1}^{\infty} \xi_k \zeta^{-k} \qquad \text{for} \quad \xi \in \ell_2^p$$

then

$$(\Gamma\eta)_- = (\Gamma_{f^\circ}\eta_+)(\zeta).$$

It was shown in [AAK1] (for the case $p=q=1$) and [Ad] (for the general case) that if the NP $(f^\circ, \mathbb{T})$ is completely indeterminate, then there is a linear fractional description of the set $\mathscr{G}(\gamma_1, \gamma_2, ...) = \mathscr{F}(f^\circ)$. In view of Lemma 3.2, this description is valid for $\Omega_0 = \mathbb{R}$ also. More precisely, the following statement is in force:

THEOREM 3.3. *Let $f^\circ \in L_\infty^{p\times q}(\Omega_0)$ and assume that the NP (f°, Ω_0) is completely indeterminate. Then*

$$\mathscr{F}(f^\circ) = T_{\mathfrak{A}}[\mathscr{S}^{p\times q}], \tag{3.6}$$

where

$$T_{\mathfrak{A}}[\mathscr{S}^{p\times q}] = \{T_{\mathfrak{A}}[\mathscr{E}]: \mathscr{E} \in \mathscr{S}^{p\times q}\}, \tag{3.7}$$

$$T_{\mathfrak{A}}[\mathscr{E}] = \{\mathfrak{a}_-(\mu)\,\mathscr{E}(\mu) + \mathfrak{b}_-(\mu)\}\{\mathfrak{b}_+(\mu)\,\mathscr{E}(\mu) + \mathfrak{a}_+(\mu)\}^{-1} \tag{3.8}$$

and the mvf

$$\mathfrak{A}(\mu)=\begin{bmatrix} \mathfrak{a}_-(\mu) & \mathfrak{b}_-(\mu) \\ \mathfrak{b}_+(\mu) & \mathfrak{a}_+(\mu) \end{bmatrix} \tag{3.9}$$

has the following properties:

1. $\mathfrak{A}(\mu)$ *is a measurable* $m \times m$ *mvf on* Ω_0 *and is* j_{pq}*-unitary a.e. on* Ω_0.

2. $\mathfrak{a}_+(\mu)$ *and* $\mathfrak{a}_-(\mu)^*$ *are the boundary values of mvf's* $\mathfrak{a}_+(\lambda)$ *and* $\mathfrak{a}_-^{\#}(\lambda)$ *that are holomorphic in* Ω_+ *and, in addition,*

$$\mathfrak{a}_+^{-1} \in \mathcal{S}_{out}^{q\times q}(\Omega_+) \qquad \text{and} \qquad (\mathfrak{a}_-^{\#})^{-1} \in \mathcal{S}_{out}^{p\times p}(\Omega_+). \tag{3.10}$$

3. *The mvf*

$$\chi(\mu) = -\mathfrak{a}_+(\mu)^{-1}\,\mathfrak{b}_+(\mu) = -\mathfrak{b}_-(\mu)^*\,[\mathfrak{a}_-(\mu)^*]^{-1} \tag{3.11}$$

is the boundary value of a mvf $\chi(\lambda) \in \mathcal{S}^{q\times p}(\Omega_+)$.

Moreover, the mvf $\mathfrak{A}(\mu)$ *which meets* (3.6) *and has properties* (1)–(3) *is unique up to a constant* j_{pq}*-unitary multiplier on the right.*

The class of mvf's $\mathfrak{A}(\mu)$ which enjoy the properties (1)–(3) will be denoted by the symbol $\mathfrak{M}(p, q)$. This class was introduced and investigated for NP $(f^\circ, \mathbb{T})$ in [Ar6]. In that paper the members of this class were called γ-generating mvf's.

We remark that if $\mathfrak{A} \in \mathfrak{M}(p, q)$, then the corresponding mvf $\chi(\lambda)$ is strictly contractive at each point $\lambda \in \Omega_+$:

$$\|\chi(\lambda)\| < 1 \qquad \text{for} \quad \lambda \in \Omega_+.$$

In fact,

$$\log\{1 - \|\chi(\mu)\|\} \in \tilde{L}_1(\Omega_0).$$

A mvf $\mathfrak{A} \in \mathfrak{M}(p, q)$ is said to be:

1. singular if $T_{\mathfrak{A}}[\mathcal{S}^{p\times q}] \subset \mathcal{S}^{p\times q}$.

2. (right) regular if $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2$ with $\mathfrak{A}_j \in \mathfrak{M}(p, q)$ for $j = 1, 2$ and $\mathfrak{A}_2$ is singular, then $\mathfrak{A}_2$ is constant.

3. strongly regular if there exists an $f \in T_{\mathfrak{A}}[\mathcal{S}^{p\times q}]$ with $\|f\|_\infty < 1$.

These three subclasses of $\mathfrak{M}(p, q)$ will be designated $\mathfrak{M}_S(p, q)$, $\mathfrak{M}_R(p, q)$ and $\mathfrak{M}_{sR}(p, q)$, respectively.

We remark that $\mathfrak{A} \in \mathfrak{M}_S(p, q)$, if and only if $\mathfrak{A}(\mu)$ is the boundary value of a mvf $W \in \mathcal{U}_S(j_{pq})$.

The next lemma was established in [Ar6] for $\Omega_0=\mathbb{T}$, but is easily extended to $\Omega_0=\mathbb{R}$.

LEMMA 3.4. *Let $\mathfrak{A}\in\mathfrak{M}(p,q)$. Then*

$$(1-\|\chi(\mu)\|)^{-1}\in\tilde{L}_1(\Omega_0) \qquad \text{if and only if} \quad \|\mathfrak{A}(\mu)\|^2\in\tilde{L}_1(\Omega_0).$$

Moreover, if either of these two conditions are in force, then $\mathfrak{A}\in\mathfrak{M}_R(p,q)$.

LEMMA 3.5. *Let $\mathfrak{A}\in\mathfrak{M}_{sR}(p,q)$, then the scalar function*

$$\|\mathfrak{A}(\mu)\|^2\in\tilde{L}_1(\Omega_0).$$

Proof. By assumption, there exists an $\mathscr{E}\in\mathscr{S}^{p\times q}$ such that the mvf

$$f=T_{\mathfrak{A}}[\mathscr{E}]$$

is strictly contractive:

$$\|f\|_\infty=\delta<1.$$

Then since $\chi\mathscr{E}\in\mathscr{S}^{q\times q}$ and $\|(\chi\mathscr{E})(\lambda)\|<1$ for every point $\lambda\in\Omega_+$, the mvf

$$c=(I_q+\chi\mathscr{E})(I_q-\chi\mathscr{E})^{-1}$$

belongs to the Carathéodory class $\mathscr{C}^{q\times q}(\Omega_+)$. Consequently,

$$\|(\Re c)(\mu)\|\in\tilde{L}_1(\Omega_0).$$

But

$$\begin{aligned}(\Re c)(\mu)&=\{I_q-(\chi\mathscr{E})(\mu)^*\}^{-1}\{I_q-(\chi\mathscr{E})(\mu)^*(\chi\mathscr{E})(\mu)\}\{I_q-(\chi\mathscr{E})(\mu)\}^{-1}\\&\geqslant\{I_q-(\chi\mathscr{E})(\mu)^*\}^{-1}\{I_q-\mathscr{E}(\mu)^*\mathscr{E}(\mu)\}\{I_q-(\chi\mathscr{E})(\mu)\}^{-1}\\&=\mathfrak{a}_+(\mu)^*\{I_q-f(\mu)^*f(\mu)\}\,\mathfrak{a}_+(\mu)\\&\geqslant(1-\delta^2)\,\mathfrak{a}_+(\mu)^*\,\mathfrak{a}_+(\mu)\end{aligned}$$

for a.e. point $\mu\in\Omega_0$. But, in view of the identity

$$\mathfrak{a}_+(\mu)^*\,\mathfrak{a}_+(\mu)=\{I_q-\chi(\mu)\,\chi(\mu)^*\}^{-1}$$

and Lemma 3.4, the asserted result is now clear. ∎

The next two theorems clarify the connection between the class $\mathfrak{M}_R(p,q)$ [resp. $\mathfrak{M}_{sR}(p,q)$] and the completely indeterminate [resp. strictly completely indeterminate] Nehari problems.

THEOREM 3.6. *Let $f^\circ \in L_\infty^{p\times q}(\Omega_0)$ with $\|f^\circ\|_\infty \leqslant 1$ and let the NP (f°, Ω_0) be completely indeterminate. Then there exists a mvf $\mathfrak{A} \in \mathfrak{M}(p, q)$ such that formula (3.6) holds. Such a mvf $\mathfrak{A}(\mu)$ is unique up to a constant j_{pq}-unitary multiplier on the right and is automatically right regular, i.e., $\mathfrak{A} \in \mathfrak{M}_R(p, q)$. If the NP (f°, Ω_0) for the given f° is strictly completely indeterminate, then $\mathfrak{A} \in \mathfrak{M}_{sR}(p, q)$.*

THEOREM 3.7. *Let $\mathfrak{A} \in \mathfrak{M}_R(p, q)$ and let $f^\circ \in T_{\mathfrak{A}}[\mathscr{S}^{p\times q}]$. Then the NP (f°, Ω_0) is completely indeterminate and*

$$\mathscr{F}(f^\circ) = T_{\mathfrak{A}}[\mathscr{S}^{p\times q}]. \tag{3.12}$$

If the given $\mathfrak{A}$ is strongly regular, then the NP (f°, Ω_0) is strictly completely indeterminate. Thus,

$$\mathfrak{A} \in \mathfrak{M}_{sR}(p, q) \Leftrightarrow \mathfrak{A} \in \mathfrak{M}_R(p, q) \qquad \text{and} \qquad \|\Gamma_{f^\circ}\| < 1.$$

The connection between the class $\mathfrak{M}_R(p, q)$ and completely indeterminate Nehari problems that is formulated in the preceding two theorems was established in [Ar6]. The statements that relate the class $\mathfrak{M}_{sR}(p, q)$ to strictly completely indeterminate Nehari problems then follow easily from the corresponding definitions.

4. THE MUCKENHOUPT CONDITION

One of the main results of this paper, Theorem 4.7, is established in this section. This theorem characterizes the class $\mathscr{U}_{sR}(j_{pq})$ of strongly regular j_{pq}-inner mvf's in terms of the matrix Muckenhoupt condition (A_2) that was used by Treil and Volberg [TrV] to characterize those weighted L_2 spaces with a matrix valued weight for which the Hilbert transform is bounded. Enroute to Theorem 4.7, we shall establish an analogous characterization for the class $\mathfrak{M}_{sR}(p, q)$; see Theorem 4.5. The proof depends essentially upon the recent result of Treil and Volberg that was referred to just above.

In order to formulate the matrix Muckenhoupt condition, we introduce the following notation for the average $A_I(\Delta)$ of a mvf Δ on a finite interval $I \subset \mathbb{R}$ of length $|I|$ such that $|I| \leqslant 2\pi$ if $\Omega_0 = \mathbb{T}$:

$$A_I(\Delta) = \begin{cases} \dfrac{1}{|I|} \displaystyle\int_I \Delta(e^{i\theta})\, d\theta & \text{if} \quad \Omega_0 = \mathbb{T} \\[2ex] \dfrac{1}{|I|} \displaystyle\int_I \Delta(\mu)\, d\mu & \text{if} \quad \Omega_0 = \mathbb{R}, \end{cases} \tag{4.1}$$

TABLE 4.1

Ω_0	$\mathbb{T}$	$\mathbb{R}$
$\mathscr{D}_+(\Delta)$	$\bigvee_{k\geqslant 0} e^{i\theta k}\mathbb{C}^n$	$\bigvee_{t\geqslant 0} \frac{e^{i\mu t}-1}{\mu}\mathbb{C}^n$
$\mathscr{D}_-(\Delta)$	$\bigvee_{k< 0} e^{i\theta k}\mathbb{C}^n$	$\bigvee_{t< 0} \frac{e^{i\mu t}-1}{\mu}\mathbb{C}^n$

whenever the integral is meaningful. Following Treil and Volberg [TrV] we shall say that an $n\times n$ positive semidefinite measurable mvf $\Delta(\mu)$ on Ω_0 satisfies the matrix Muckenhoupt condition (A_2) if

$$\sup_I \|A_I(\Delta)^{1/2}\, A_I(\Delta^{-1})^{1/2}\| < \infty. \tag{4.2}$$

LEMMA 4.1. *If $\Delta(\mu)$ satisfies condition* (4.2), *then*

$$\|\Delta(\mu)^{\pm 1}\| \in \tilde{L}_1(\Omega_0). \tag{4.3}$$

Proof. For $\Omega_0=\mathbb{T}$, this conclusion is immediate from (4.2), just take $I=[0, 2\pi]$. For $\Omega_0=\mathbb{R}$, this conclusion is a consequence of Lemma 2.2 in [TrV]. ▮

Let $\Delta(\mu)$ be an $n\times n$ measurable mvf on Ω_0 which is positive semidefinite a.e. on Ω_0 and let $L_2^n(\Delta)$ denote the Hilbert space of $n\times 1$ measurable vvf's $f(\mu)$ on Ω_0 with norm

$$\|f\|_\Delta = \|\Delta^{1/2} f\|_{st}, \tag{4.4}$$

where $\|\ \|_{st}$ denotes the norm based on the standard inner product in $L_2^n(\Omega_0)$. Next, for any family $\mathscr{L}_t$, $t\in\mathbb{T}$, of sets in $L_2^n(\Delta)$, let

$$\bigvee_{t\in T} \mathscr{L}_t$$

denote the closed linear span of the designated sets in $L_2^n(\Delta)$. Suppose further that

$$\Delta \in \widetilde{L_1^{n\times n}}(\Omega_0) \tag{4.5}$$

so that the subspaces $\mathscr{D}_\pm(\Delta)$ that will be introduced in Table 4.1 are well defined.

For any closed subspace L of $L_2^n(\Delta)$, let P_L denote the orthogonal projection from $L_2^n(\Delta)$ onto L and let $A|_L$ denote the restriction of the operator A to L.

We now formulate the result of Treil and Volberg [TrV] in a form that will be convenient for our needs.

THEOREM 4.2 (Treil and Volberg [TrV]). *Let $\Delta(\mu)$ be a measurable $n\times n$ mvf on Ω_0 which is positive definite for a.e. $\mu\in\Omega_0$. Then $\Delta(\mu)$ meets the matrix Muckenhoupt condition* (4.2) *if and only if*

$$\Delta\in \widetilde{L_1^{n\times n}}(\Omega_0) \qquad \textit{and} \qquad \|P_{\mathscr{D}_-(\Delta)}|_{\mathscr{D}_+(\Delta)}\|<1. \tag{4.6}$$

The second condition in (4.6) states that the angle between the "past" $\mathscr{D}_-(\Delta)$ and the "future" $\mathscr{D}_+(\Delta)$ in the Hilbert space $L_2^n(\Delta)$ is strictly positive.

THEOREM 4.3. *Let $\Delta\in \widetilde{L_1^{n\times n}}(\Omega_0)$ be positive definite a.e. on Ω_0. Then the following are equivalent*:

1. *There exists a solution $\psi_-(\mu)$ of the factorization problem*

$$\Delta(\mu)=\psi_-(\mu)^*\,\psi_-(\mu) \text{ a.e. on } \Omega_0, \tag{4.7}$$

where $\psi_-^\#$ is an outer mvf in the Smirnov class $\mathscr{N}_+^{n\times n}(\Omega_+)$.

2. *There exists a solution $\psi_+(\mu)$ of the factorization problem*

$$\Delta(\mu)=\psi_+(\mu)^*\,\psi_+(\mu) \text{ a.e. on } \Omega_0, \tag{4.8}$$

where ψ_+ is an outer mvf in the Smirnov class $\mathscr{N}_+^{n\times n}(\Omega_+)$.

3.

$$\log\{\det\Delta(\mu)\}\in\tilde{L}_1(\Omega_0). \tag{4.9}$$

If any one (*and hence all three*) *of these conditions hold, then the factors $\psi_-(\mu)$ and $\psi_+(\mu)$ are unique up to a left constant unitary $n\times n$ matrix multiplier. Moreover,*

$$\frac{\psi_-^\#}{\rho_\omega}\in H_2^{n\times n}(\Omega_+) \qquad \textit{and} \qquad \frac{\psi_+}{\rho_\omega}\in H_2^{n\times n}(\Omega_+) \tag{4.10}$$

for every point $\omega\in\Omega_+$.

Theorem 4.3 is well known and was obtained by V. N. Zasukhin [Za] and M. G. Krein (see Chapter 2 of Rozanov [Ro] and the notes to the chapter on page 200 for the history and additional references) and independently by N. Wiener [Wi] some forty odd years ago.

LEMMA 4.4. *Let* $\Delta \in \widetilde{L_1^{n\times n}}(\Omega_0)$ *be positive definite a.e. on* Ω_0 *and suppose that* $\Delta(\mu)$ *satisfies at least one (and hence all three) of the conditions* (4.7)–(4.9) *in Theorem* 4.3. *Let* $\psi_-^\#$ *and* ψ_+ *be the essentially unique outer mvf's considered in Theorem* 4.3. *Then*

1. *The mvf*

$$g = \psi_- \psi_+^{-1} \tag{4.11}$$

is unitary a.e. on Ω_0.

2. *The norm of the Hankel operator*

$$\Gamma_g = P_{K_2^n} M_g |_{H_2^n} \tag{4.12}$$

is equal to the norm of the restricted projection considered in Theorem 4.2:

$$\|\Gamma_g\| = \|P_{\mathscr{D}_-(\Delta)}|_{\mathscr{D}_+(\Delta)}\|. \tag{4.13}$$

Proof. The first assertion is immediate from formulas (4.7) and (4.9). To verify the second assertion, we first observe that

$$f_+ \in \mathscr{D}_+(\Delta) \qquad \text{if and only if} \quad \psi_+ f_+ \in H_2^n(\Omega_0),$$

whereas

$$f_- \in \mathscr{D}_-(\Delta) \qquad \text{if and only if} \quad \psi_- f_- \in K_2^n(\Omega_0).$$

Thus, as

$$\langle f_+, f_- \rangle_\Delta = \langle \Delta f_+, f_- \rangle_{st},$$

where $\langle\ ,\ \rangle_{st}$ denotes the standard inner product in $L_2^n(\Omega_0)$, we see that

$$\|P_{\mathscr{D}_-} f_+\|_\Delta^2 = \sup\{|\langle \Delta f_+, f_- \rangle_{st}| : f_- \in \mathscr{D}_-(\Delta) \text{ and } \|f_-\|_\Delta = 1\},$$

while

$$\begin{aligned} \langle \Delta f_+, f_- \rangle_{st} &= \langle \psi_-^* \psi_- \psi_+^{-1} \psi_+ f_+, f_- \rangle_{st} \\ &= \langle g\psi_+ f_+, \psi_- f_- \rangle_{st} \\ &= \langle g h_+, h_- \rangle_{st}, \end{aligned}$$

where $h_+ = \psi_+ f_+$ belongs to $H_2^n(\Omega_0)$, $h_- = \psi_- f_-$ belongs to $K_2^n(\Omega_0)$, and

$$\|h_\pm\|_{st} = \|f_\pm\|_\Delta.$$

Thus,

$$\|P_{\mathscr{D}_-} f_+\|_\Delta^2 = \sup\{|\langle g h_+, h_-\rangle_{st}| : h \in K_2^n \text{ and } \|h_-\|_{st} = 1\}$$

and hence the desired result (4.13) now follows by standard arguments. ▮

THEOREM 4.5. *Let* $\mathfrak{A} \in \mathfrak{M}(p, q)$, *let*

$$\mathscr{E} = \begin{cases} \begin{bmatrix} 0_{r\times q} \\ I_q \end{bmatrix} & \text{if } \; p > q \\ I_p & \text{if } \; p = q \\ [I_p \quad 0_{p\times r}] & \text{if } \; q > p, \end{cases} \tag{4.14}$$

where $r = |p - q|$, *let*

$$\varphi_-^\varepsilon(\mu) = \mathfrak{a}_-(\mu) + \mathfrak{b}_-(\mu)\,\mathscr{E}^* \quad \text{and} \quad \varphi_+^\varepsilon(\mu) = \mathfrak{b}_+(\mu)\,\mathscr{E} + \mathfrak{a}_+(\mu) \tag{4.15}$$

and let

$$\Delta(\mu) = \begin{cases} \varphi_-^\varepsilon(\mu)^*\, \varphi_-^\varepsilon(\mu) & \text{if } \; p \geqslant q \\ \varphi_+^\varepsilon(\mu)^*\, \varphi_+^\varepsilon(\mu) & \text{if } \; q \geqslant p. \end{cases} \tag{4.16}$$

Then $\mathfrak{A}(\mu)$ *is strongly regular if and only if* $\Delta(\mu)$ *satisfies the matricial Muckenhoupt condition* (4.2) *and* $\mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0)$.

It is convenient to establish the following lemma before starting the proof of the theorem.

LEMMA 4.6. *Let* $\mathfrak{A} \in \mathfrak{M}(p, q)$ *with* $p \geqslant q$ *and let* $\varphi_\pm^\varepsilon(\mu)$ *be defined by formula* (4.15). *Then*:

1. $(\varphi_-^\varepsilon)^\#$ *and* φ_+^ε *are outer mvf's in the Smirnov classes* $\mathscr{N}_+^{p\times p}(\Omega_+)$ *and* $\mathscr{N}_+^{q\times q}(\Omega_+)$, *respectively.*

2.

$$(\rho_\omega(\varphi_-^\varepsilon)^\#)^{-1} \in H_2^{p\times p}(\Omega_+) \qquad \text{and} \qquad (\rho_\omega \varphi_+^\varepsilon)^{-1} \in H_2^{q\times q}(\Omega_+) \tag{4.17}$$

for every point $\omega \in \Omega_+$.

If $\mathfrak{A} \in \mathfrak{M}_{sR}(p, q)$, *then we also have*

3.

$$(\rho_\omega^{-1}(\varphi_-^\varepsilon)^\#) \in H_2^{p\times p}(\Omega_+) \qquad \text{and} \qquad (\rho_\omega^{-1} \varphi_+^\varepsilon) \in H_2^{q\times q}(\Omega_+) \tag{4.18}$$

for every point $\omega \in \Omega_+$.

Proof. The first two statements are established in [Ar6] for $\Omega_+ = \mathbb{D}$, but are easily extended via (3.5) to $\Omega_+ = \mathbb{C}_+$.

Next, if $\mathfrak{A} \in \mathfrak{M}_{sR}(p, q)$, then

$$\|\varphi_\pm^\varepsilon(\mu)\|^2 \in \widetilde{L}_1(\Omega_0),$$

by Lemma 3.5. Therefore, in view of statement (1), statement (3) now follows from the Smirnov maximum principle. For more information on the latter, see e.g., the review paper of Katsnelson and Kirstein [KK]. ▌

Proof of Theorem 4.5. There are three cases to consider: $p = q$, $p > q$ and $p < q$. We begin with the case $p = q$, because the other two cases are established by transforming them to a version of the first case. The same embedding technique was used for other purposes in [AFK].

Case 1. $p = q$: Suppose first that $\mathfrak{A} \in \mathfrak{M}_{sR}(p, p)$ and let

$$f_{I_p} = T_{\mathfrak{A}}[I_p] = \varphi_- \varphi_+^{-1},$$

where, for short, we have set $\varphi_\pm(\mu) = \varphi_\pm^\varepsilon(\mu)$ with $\mathscr{E} = I_p$. Then, by Lemma 3.5,

$$\Delta(\mu) = \varphi_-(\mu)^* \varphi_-(\mu)$$

belongs to $\widetilde{L_1^{p\times p}}(\Omega_0)$. Moreover, by Theorem 3.7, the set of solutions to the NP (f_{I_p}, Ω_0) is given by formula (3.12). Thus,

$$\Gamma_f = \Gamma_{f_{I_p}}$$

for every

$$f \in T_{\mathfrak{A}}[\mathscr{S}^{p\times q}]$$

and, because $\mathfrak{A} \in \mathfrak{M}_{sR}(p, p)$, at least one of these mvf's f has $\|f\|_\infty < 1$. This ensures that

$$\|\Gamma_{f_{I_p}}\| < 1$$

and hence, by Theorem 4.2 and Lemma 4.4, that the $p \times p$ mvf

$$\Delta(\mu) = \varphi_-(\mu)^* \varphi_-(\mu) = \varphi_+(\mu)^* \varphi_+(\mu) \tag{4.19}$$

meets the Muckenhoupt condition (4.2).

Suppose next that $\Delta(\mu)$ meets the matricial Muckenhoupt condition (4.2) and that $\mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0)$, where in the present case $m = 2p$. Then, by Lemma 3.4, $\mathfrak{A} \in \mathfrak{M}_R(p, p)$ and, by another application of Theorem 4.2 and

Lemma 4.4, it also follows that $\Delta \in \widetilde{L_1^{p\times p}}(\Omega_0)$ and $\|\Gamma_{f_{I_p}}\| < 1$. Therefore, by Lemma 3.1, the NP (f_{I_p}, Ω_0) is strictly completely indeterminate and hence, by Theorem 3.6, $\mathfrak{A}(\mu)$ is strongly regular. This completes the proof of Case 1.

Case 2. $p > q$: Let

$$\mathfrak{A}^\circ(\mu) = \begin{bmatrix} \mathfrak{a}_-^\circ(\mu) & \mathfrak{b}_-^\circ(\mu) \\ \mathfrak{b}_+^\circ(\mu) & \mathfrak{a}_+^\circ(\mu) \end{bmatrix} \tag{4.20}$$

be the $2p \times 2p$ mvf with $p \times p$ block entries that are defined in terms of the block entries of $\mathfrak{A}(\mu)$ by the formulas

$$\begin{aligned} \mathfrak{a}_-^\circ(\mu) &= \mathfrak{a}_-(\mu), & \mathfrak{b}_-^\circ(\mu) &= [0_{p\times r} \quad \mathfrak{b}_-(\mu)], \\ \mathfrak{b}_+^\circ(\mu) &= \begin{bmatrix} 0_{r\times p} \\ b_+(\mu) \end{bmatrix}, & \mathfrak{a}_+^\circ(\mu) &= \begin{bmatrix} I_r & 0_{r\times q} \\ 0_{q\times r} & a_+(\mu) \end{bmatrix}, \end{aligned} \tag{4.21}$$

where $r = p - q$ and let

$$\varphi_-^\circ(\mu) = \mathfrak{a}_-^\circ(\mu) + \mathfrak{b}_-^\circ(\mu) \qquad \text{and} \qquad \varphi_+^\circ(\mu) = \mathfrak{b}_+^\circ(\mu) + \mathfrak{a}_+^\circ(\mu). \tag{4.22}$$

Then it is readily checked that $\mathfrak{A}^\circ \in \mathfrak{M}(p, p)$ and that

$$\varphi_-^\circ(\mu) = \varphi_-^\varepsilon(\mu).$$

Therefore,

$$\Delta(\mu) = \varphi_-^\varepsilon(\mu)^* \, \varphi_-^\varepsilon(\mu) = \varphi_-^\circ(\mu)^* \, \varphi_-^\circ(\mu)$$

a.e. on Ω_0, and, since

$$T_{\mathfrak{A}^\circ}[I_p] = \varphi_-^\circ(\varphi_+^\circ)^{-1}$$

is unitary a.e. on Ω_0, we also have

$$\Delta(\mu) = \varphi_+^\circ(\mu)^* \, \varphi_+^\circ(\mu)$$

a.e. on Ω_0. The rest of the proof is divided into steps.

Step 1. If $p > q$, then the following three sets of conditions are equivalent:

1. $\mathfrak{A}^\circ \in \mathfrak{M}_{sR}(p, p)$.
2. Δ meets the matricial Muckenhoupt condition (4.2) and $\mathfrak{A}^\circ \in \widetilde{L_2^{2p\times 2p}}(\Omega_0)$.
3. $\mathfrak{A}^\circ \in \widetilde{L_2^{2p\times 2p}}(\Omega_0)$ and the Hankel operator $\Gamma_{f_{I_p}}^\circ$ with symbol $f_{I_p} = \varphi_-^\circ(\varphi_+^\circ)^{-1}$ is strictly contractive.

Proof of Step 1. Since $\mathfrak{A}^\circ \in \mathfrak{M}(p, p)$, it is covered by Case 1.

Step 2. Let $\mathscr{E}^\circ = [0_{p\times r} \;\; \mathscr{E}]$, where $\mathscr{E}$ is given by formula (4.14) and $r = p - q$, and let

$$g^\circ = T_{\mathfrak{A}^\circ}[\mathscr{E}^\circ] \qquad \text{and} \qquad g = T_{\mathfrak{A}}[\mathscr{E}]. \tag{4.23}$$

Then the Hankel operator Γ_{g° with symbol g° has the same norm as the Hankel operator Γ_g with symbol g:

$$\|\Gamma^\circ_{g^\circ}\| = \|\Gamma_g\|. \tag{4.24}$$

Proof of Step 2. It is readily checked that

$$g^\circ = [0_{p\times r} \quad g].$$

The rest is immediate from the definitions.

Step 3. If $p > q$ and $\mathfrak{A}^\circ \in \mathfrak{M}_{sR}(p, p)$, then $\mathfrak{A} \in \mathfrak{M}_{sR}(p, q)$.

Proof of Step 3. The proof is based on the following sequence of implications:

$$\begin{aligned}
&\mathfrak{A}^\circ \in \mathfrak{M}_{sR}(p, p) \\
&\qquad \Rightarrow \mathfrak{A}^\circ \in \widetilde{L_2^{p\times 2p}}(\Omega_0) && \text{(by Lemma 3.5)} \\
&\qquad \Rightarrow \mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0) && \text{(by formula (4.20))} \\
&\qquad \Rightarrow \mathfrak{A} \in \mathfrak{M}_R(p, q) && \text{(by Lemma 3.4)} \\
&\qquad \Rightarrow T_{\mathfrak{A}}[\mathscr{S}^{p\times q}] = \mathscr{F}(g), && \text{where} \quad g = T_{\mathfrak{A}}[\mathscr{E}] \quad \text{(by definition)}, \\
&\qquad \Rightarrow \exists f \in T_{\mathfrak{A}}[\mathscr{S}^{p\times q}] && \text{with} \quad \|f\|_\infty < 1 \quad (\text{since } \|\Gamma_g\| = \|\Gamma^\circ_{g^\circ}\| < 1) \\
&\qquad \Rightarrow \mathfrak{A} \in \mathfrak{M}_{sR}(p, q).
\end{aligned}$$

Step 4. If $p > q$ and $\mathfrak{A} \in \mathfrak{M}_{sR}(p, q)$, then $\mathfrak{A}^\circ \in \mathfrak{M}_{sR}(p, p)$.

Proof of Step 4. To begin with, the assumption that $\mathfrak{A} \in \mathfrak{M}_{sR}(p, q)$ guarantees the existence of an $f \in T_{\mathfrak{A}}[\mathscr{S}^{p\times q}]$ with $\|f\|_\infty < 1$. Therefore, the corresponding Hankel operator Γ_f is also strictly contractive. Thus, as

$$\Gamma_g = \Gamma_f \qquad \text{and} \qquad \|\Gamma_g\| = \|\Gamma^\circ_{g^\circ}\|,$$

we obtain the bound

$$\|\Gamma^\circ_{g^\circ}\| < 1. \tag{4.25}$$

At the same time,

$$\begin{aligned}
\mathfrak{A} \in \mathfrak{M}_{sR}(p, q) &\Rightarrow \mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0) && \text{(by Lemma 3.5)}\\
&\Rightarrow \mathfrak{A}^\circ \in \widetilde{L_2^{2p\times 2p}}(\Omega_0) && \text{(by formula (4.20))}\\
&\Rightarrow \mathfrak{A}^\circ \in \mathfrak{M}_R(p, p) && \text{(by Lemma 3.4).}
\end{aligned}$$

The latter conclusion guarantees that

$$T_{\mathfrak{A}^\circ}[\mathscr{S}^{p\times p}] = \mathscr{F}(g^\circ)$$

and hence, in view of the bound (4.25), there exists an $f^\circ \in T_{\mathfrak{A}^\circ}[\mathscr{S}^{p\times p}]$ with $\|f^\circ\| < 1$. Therefore, $\mathfrak{A}^\circ \in \mathfrak{M}_{sR}(p, p)$, as claimed. This completes the proof of the step, and, thanks to the preceding step, the proof of the theorem for the case $p > q$.

Case 3. $q > p$: Let $\mathfrak{A}^\circ(\mu)$ denote the $2q \times 2q$ mvf with $q \times q$ blocks that are defined by the formulas

$$\mathfrak{a}_-^\circ(\mu) = \begin{bmatrix} a_-(\mu) & 0_{p\times r} \\ 0_{r\times p} & I_r \end{bmatrix}, \qquad \mathfrak{b}_-^\circ(\mu) = \begin{bmatrix} b_-(\mu) \\ 0_{r\times q} \end{bmatrix}.$$

$$\mathfrak{b}_+^\circ(\mu) = [b_+(\mu) \quad 0_{q\times r}] \qquad \text{and} \qquad \mathfrak{a}_+^\circ(\mu) = a_+(\mu).$$

It is readily checked that $\mathfrak{A}^\circ(\mu)$ belongs to the class $\mathfrak{M}(q, q)$ and that

$$\varphi_+^\varepsilon(\mu) = \varphi_+^\circ(\mu).$$

The rest of the proof amounts to first invoking Case 1 to obtain

$$\mathfrak{A}^\circ \in \mathfrak{M}_{sR}(q, q) \Leftrightarrow \mathfrak{A}^\circ \in \widetilde{L_2^{2q\times 2q}}\ (\Omega_0) \text{ and } \Delta(\mu)$$

meets the matricial Muckenhoupt condition (4.2).

Then, since

$$\mathfrak{A}^\circ \in \widetilde{L_2^{2q\times 2q}}\ (\Omega_0) \Leftrightarrow \mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0),$$

the proof is completed much as in the verification of Case 2 by showing that

$$\mathfrak{A}^\circ \in \mathfrak{M}_{sR}(q, q) \Leftrightarrow \mathfrak{A} \in \mathfrak{M}_{sR}(p, q).$$

We omit the details. ▮

We turn next to one of the main theorems of this paper. To this end, let $W \in \mathcal{U}(j_{pq})$, let $\mathcal{E}$ be defined as in (4.14) with $r = |p-q|$, let

$$\psi_-^{\varepsilon}(\mu) = w_{11}(\mu) + w_{12}(\mu)\, \mathcal{E}^* \quad \text{and} \quad \psi_+^{\varepsilon}(\mu) = w_{21}(\mu)\, \mathcal{E} + w_{22}(\mu). \tag{4.26}$$

THEOREM 4.7. *Let $W \in \mathcal{U}(j_{pq})$ and let $\psi_\pm^{\varepsilon}$ be defined in terms of the block decomposition of W by formulas* (4.14) *and* (4.26). *Then $W(\lambda)$ is strongly regular if and only if $W \in \widetilde{L_2^{m\times m}}(\Omega_0)$ and the mvf that is defined by the formula*

$$\Delta(\mu) = \begin{cases} \psi_-^{\varepsilon}(\mu)^*\, \psi_-^{\varepsilon}(\mu) & \text{if} \quad p \geqslant q \\ \psi_+^{\varepsilon}(\mu)^*\, \psi_+^{\varepsilon}(\mu) & \text{if} \quad q \geqslant p \end{cases} \tag{4.27}$$

meets the matricial Muckenhoupt condition (4.2).

Proof. Let $W \in \mathcal{U}(j_{pq})$. Then

$$W(\mu) = \begin{bmatrix} b_1(\mu) & 0 \\ 0 & b_2(\mu)^{-1} \end{bmatrix} \mathfrak{A}(\mu)$$

a.e. on Ω_0, where $b_1 \in \mathcal{S}_{in}^{p\times p}(\Omega_+)$, $b_2 \in \mathcal{S}_{in}^{q\times q}(\Omega_+)$ and $\mathfrak{A} \in \mathfrak{M}(p, q)$. Thus,

$$\psi_-^{\varepsilon}(\mu) = b_1(\mu)\, \varphi_-^{\varepsilon}(\mu) \qquad \text{and} \qquad \psi_+^{\varepsilon}(\mu) = b_2(\mu)^{-1}\, \varphi_+^{\varepsilon}(\mu)$$

for a.e. $\mu \in \Omega_0$ and hence the formulas for $\Delta(\mu)$ given in (4.16) and (4.27) agree a.e. on Ω_0. Moreover, since

$$T_W[\mathcal{E}] = b_1\, T_{\mathfrak{A}}[\mathcal{E}]\, b_2$$

for every $\mathcal{E} \in \mathcal{S}^{p\times q}(\Omega_+)$, it is readily checked that $W \in \mathcal{U}_{sR}(j_{pq})$ if and only if $\mathfrak{A} \in \mathfrak{M}_{sR}(p, q)$. The asserted equivalence is therefore immediate from Theorem 4.5. ∎

4.1. *Another version of Theorem* 4.5. If $\mathfrak{A} \in \mathfrak{M}(p, q)$ and the weight $\Delta(\mu)$ that is defined by (4.16) satisfies the matricial Muckenhoupt condition (4.2), then the entries in $\varphi_\pm^{\varepsilon}(\mu)$ belong $\tilde{L}_2(\Omega_0)$. This is not enough to ensure that the full matrix $\mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0)$ (which in turn guarantees that $\mathfrak{A} \in \mathfrak{M}_R(p, q)$ via Lemma 3.4). However, it is known for example that if

$$(\mathfrak{a}_- \pm \mathfrak{b}_- \mathcal{E}^*) \in \widetilde{L_2^{p\times p}}\,(\Omega_0) \tag{4.28}$$

for at least one isometric matrix $\mathcal{E}$, then

$$(\mathfrak{b}_+ \mathcal{E} \pm \mathfrak{a}_+)^*\, (\mathfrak{b}_+ \mathcal{E} \pm \mathfrak{a}_+) = \mathcal{E}^*(\mathfrak{a}_- \pm \mathfrak{b}_- \mathcal{E}^*)^*\, (\mathfrak{a}_- \pm \mathfrak{b}_- \mathcal{E}^*)\, \mathcal{E} \tag{4.29}$$

(which is valid a.e. on Ω_0) guarantees that

$$(\mathfrak{b}_+\mathscr{E} \pm \mathfrak{a}_+) \in \widetilde{L_2^{q\times q}}(\Omega_0) \tag{4.30}$$

and hence that

$$\mathfrak{a}_- \in \widetilde{L_2^{p\times p}}(\Omega_0) \qquad \text{and} \qquad \mathfrak{a}_+ \in \widetilde{L_2^{q\times q}}(\Omega_0).$$

The relations

$$\mathfrak{a}_-(\mu)\,\mathfrak{a}_-(\mu)^* = I_p + \mathfrak{b}_-(\mu)\,\mathfrak{b}_-(\mu)^* \geqslant \mathfrak{b}_-(\mu)\,\mathfrak{b}_-(\mu)^*$$

and

$$\mathfrak{a}_+(\mu)\,\mathfrak{a}_+(\mu)^* = I_q + \mathfrak{b}_+(\mu)\,\mathfrak{b}_+(\mu)^* \geqslant \mathfrak{b}_+(\mu)\,\mathfrak{b}_+(\mu)^*$$

which are valid a.e. on Ω_0, then serve to show that the assumption (4.28) guarantees that $\mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0)$ when $p \geqslant q$.

A similar argument based on the identity

$$(\mathfrak{a}_- \pm \mathfrak{b}_-\mathscr{E}^*)^*\,(\mathfrak{a}_- \pm \mathfrak{b}_-\mathscr{E}^*) = \mathscr{E}(\mathfrak{a}_+ \pm \mathfrak{b}_+\mathscr{E})^*\,(\mathfrak{a}_+ \pm \mathfrak{b}_+\mathscr{E})\,\mathscr{E}^*, \tag{4.31}$$

which is valid a.e. on Ω_0 for any coisometric matrix $\mathscr{E}$, shows that if (4.30) holds for at least one coisometric matrix $\mathscr{E}$, then $\mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0)$ when $q \geqslant p$.

These observations enable us to extract most of the following theorem from Theorem 4.5.

THEOREM 4.8. *Let $\mathfrak{A} \in \mathfrak{M}(p, q)$, let $\varphi_\pm^{\mathscr{E}}$ be defined by formula* (4.15) *and let $\Delta_{\mathscr{E}}$ be defined by formula* (4.16) *where now $\mathscr{E} \in \mathbb{C}^{p\times q}$ is any isometric matrix if $p \geqslant q$ and any coisometric matrix if $p \leqslant q$. Then the following conditions are equivalent*:

1. $\mathfrak{A} \in \mathfrak{M}_{sR}(p, q)$.

2. *$\Delta_{\mathscr{E}}$ and $\Delta_{-\mathscr{E}}$ satisfy the matricial Muckenhoupt condition* (4.2) *for at least one of the considered $\mathscr{E}$'s.*

3. *$\Delta_{\mathscr{E}}$ satisfies the matricial Muckenhoupt condition* (4.2) *for every one of the considered $\mathscr{E}$'s.*

4. *$\mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0)$ and $\Delta_{\mathscr{E}}$ satisfy the matricial Muckenhoupt condition for at least one of the considered $\mathscr{E}$'s.*

5. *$\mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0)$ and $\|\Gamma_f\| < 1$ for some $f \in T_{\mathfrak{A}}[\mathscr{S}^{p\times q}]$.*

6. *$\mathfrak{A} \in \mathfrak{M}_R(p, q)$ and $\|\Gamma_f\| < 1$ for some $f \in T_{\mathfrak{A}}[\mathscr{S}^{p\times q}]$.*

Proof. Suppose first that $p \geqslant q$ and let

$$\mathscr{E}_0 = \begin{bmatrix} 0_{r\times q} \\ I_q \end{bmatrix}, \qquad r = p - q.$$

Then, in terms of the current notation, Theorem 4.5 states that

$$\mathfrak{A} \in \mathfrak{M}_{sR}(p, q) \Leftrightarrow \mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0) \text{ and } \Delta_{\mathscr{E}_0} \text{ satisfies (4.2).}$$

It is convenient to let

$$\mathfrak{A}_V = \begin{bmatrix} \mathfrak{a}_- V & \mathfrak{b}_- \\ \mathfrak{b}_+ V & \mathfrak{a}_+ \end{bmatrix}$$

and

$$\Delta^V = (\mathfrak{a}_- V + \mathfrak{b}_- \mathscr{E}_0^*)^* \, (\mathfrak{a}_- V + \mathfrak{b}_- \mathscr{E}_0^*),$$

where $V \in \mathbb{C}^{p\times p}$ is unitary. Then, clearly

$$\mathfrak{A} \in \mathfrak{M}(p, q) \Leftrightarrow \mathfrak{A}_V \in \mathfrak{M}(p, q)$$

and, since

$$T_{\mathfrak{A}}[\mathscr{S}^{p\times q}] = T_{\mathfrak{A}_V}[\mathscr{S}^{p\times q}],$$

it follows further that

$$\mathfrak{A} \in \mathfrak{M}_{sR}(p, q) \Leftrightarrow \mathfrak{A}_V \in \mathfrak{M}_{sR}(p, q). \tag{4.32}$$

Now, for any given $p \times q$ isometric matrix $\mathscr{E}$, let V be a unitary matrix such that

$$V\mathscr{E}_0 = \mathscr{E}.$$

Then it is readily checked that

$$\Delta_{\mathscr{E}} = V\Delta^V V^* \tag{4.33}$$

and hence that

$$\Delta_{\mathscr{E}} \text{ satisfies (4.2)} \Leftrightarrow \Delta^V \text{ satisfies (4.2).} \tag{4.34}$$

The proof of the theorem for $p \geqslant q$ is easily completed as follows:

(1) $\Rightarrow$ (3): Clearly (1) implies that $\mathfrak{A}_V \in \mathfrak{M}_{sR}(p, q)$ and hence, by Theorem 4.5, Δ^V satisfies the matricial Muckenhoupt condition (4.2). Therefore, $\Delta_{\mathscr{E}}$ also satisfies this condition by formula (4.33).

(3) ⇒ (2) ⇒ (4): The first implication is selfevident. The second follows from the discussion preceding the statement of the theorem.

(4) ⇒ (1): Clearly (4) implies that $\mathfrak{A}_V \in \widetilde{L_2^{m\times m}}(\Omega_0)$, and, by formula (4.33), that Δ^V satisfies the matricial Muckenhoupt condition (4.2). Therefore $\mathfrak{A}_V \in \mathfrak{M}_{sR}(p, q)$ and, as we have already noted in (4.32), this is equivalent to (1).

The proof of the equivalence of (1)–(4) for $q > p$ is similar and is omitted. The equivalence of (5) and (1) rests on Theorems 4.5 and 4.2 and Lemma 4.4. The assumption that $\mathfrak{A} \in \widetilde{L_2^{m\times m}}(\Omega_0)$ guarantees that $\Delta \in \widetilde{L_1^{p\times p}}(\Omega_0)$ [resp. $\widetilde{L_1^{q\times q}}(\Omega_0)$] if $p \geqslant q$ [resp. $q \geqslant p$]. The equivalence of (1) and (6) is supplied by Theorem 3.7. ∎

5. PARAMETRIZATION AND CHARACTERIZATIONS OF MVF'S FROM $\mathfrak{M}(p, p)$, $\mathfrak{M}_R(p, p)$ AND $\mathfrak{M}_{sR}(p, p)$, AND AN EXAMPLE

5.1. *Parametrization of mvf's* $\mathfrak{A} \in \mathfrak{M}(p, q)$. Every mvf $\mathfrak{A} \in \mathfrak{M}(p, q)$ can be parametrized in terms of a mvf $\chi \in \mathcal{S}^{q\times p}(\Omega_+)$ which satisfies the condition

$$\log \det\{I_q - \chi(\mu)\, \chi(\mu)^*\} \in \tilde{L}_1(\Omega_0) \tag{5.1}$$

or, equivalently, the condition

$$\log\{1 - \|\chi(\mu)\|\} \in \tilde{L}_1(\Omega_0). \tag{5.2}$$

Indeed, for each such mvf $\chi(\mu)$, there exist essentially unique solutions $\mathfrak{a}_-(\mu)$ and $\mathfrak{a}_+(\mu)$ to the factorization problems

$$\mathfrak{a}_+(\mu)^*\, \mathfrak{a}_+(\mu) = \{I_q - \chi(\mu)\, \chi(\mu)^*\}^{-1} \tag{5.3}$$

and

$$\mathfrak{a}_-(\mu)^*\, \mathfrak{a}_-(\mu) = \{I_p - \chi(\mu)^*\, \chi(\mu)\}^{-1} \tag{5.4}$$

a.e. on Ω_0 such that

$$\mathfrak{a}_+(\lambda)^{-1} \in \mathcal{S}_{out}^{q\times q}(\Omega_+) \qquad \text{and} \qquad (\mathfrak{a}_-^{\#}(\lambda))^{-1} \in \mathcal{S}_{out}^{p\times p}. \tag{5.5}$$

Then

$$\mathfrak{A}(\mu) = \begin{bmatrix} \mathfrak{a}_-(\mu) & -\mathfrak{a}_-(\mu)\, \chi(\mu)^* \\ -\mathfrak{a}+(\mu)\, \chi(\mu) & \mathfrak{a}_+(\mu) \end{bmatrix} \in \mathfrak{M}(p, q). \tag{5.6}$$

Conversely, if

$$\mathfrak{A}(\mu) = \begin{bmatrix} \mathfrak{a}_-(\mu) & \mathfrak{b}_-(\mu) \\ \mathfrak{b}_+(\mu) & \mathfrak{a}_+(\mu) \end{bmatrix} \in \mathfrak{M}(p, q), \tag{5.7}$$

then

$$\chi = -\mathfrak{a}_+^{-1} \mathfrak{b}_+ = -\mathfrak{b}_-^{\#} (\mathfrak{a}_-^{\#})^{-1} \tag{5.8}$$

belongs to $\mathscr{S}^{q \times p}(\Omega_+)$ and satisfies the condition (5.1) (and so too (5.2)). Moreover, $\mathfrak{a}_+(\lambda)$ and $\mathfrak{a}_-(\lambda)$ are solutions of the factorization problems (5.3) and (5.4), respectively, and satisfy (5.5). Thus, the parametrization formula (5.6) holds. Such parametrization formulas were considered in [Ar6].

5.2. *Another parametrization of mvf's* $\mathfrak{A} \in \mathfrak{M}(p, p)$. If $p = q$, then another useful parametrization is obtained from (5.6) by setting

$$c(\lambda) = \{I_p + \chi(\lambda)\}\{I_p - \chi(\lambda)\}^{-1}. \tag{5.9}$$

The mvf $c(\lambda)$ belongs to the Carathéodory class $\mathscr{C}^{p \times p}(\Omega_+)$ and hence

$$\frac{c(\mu) + c(\mu)^*}{2} \in \widetilde{L_1^{p \times p}}(\Omega_0). \tag{5.10}$$

Moreover, since

$$\begin{aligned} \frac{c(\mu) + c(\mu)^*}{2} &= \{I_p - \chi(\mu)\}^{-1} \{I_p - \chi(\mu)\,\chi(\mu)^*\}\{I_p - \chi(\mu)^*\}^{-1} \\ &= \{I_p - \chi(\mu)^*\}^{-1} \{I_p - \chi(\mu)^*\,\chi(\mu)\}\{I_p - \chi(\mu)\}^{-1} \end{aligned} \tag{5.11}$$

a.e. on Ω_0, the Szegö condition (5.1) is equivalent to the requirement that

$$\log \det \left\{ \frac{c(\mu) + c(\mu)^*}{2} \right\} \in \tilde{L}_1(\Omega_0). \tag{5.12}$$

THEOREM 5.1. *Let*

$$\mathfrak{A}(\mu) = \begin{bmatrix} \mathfrak{a}_-(\mu) & \mathfrak{b}_-(\mu) \\ \mathfrak{b}_+(\mu) & \mathfrak{a}_+(\mu) \end{bmatrix} \in \mathfrak{M}(p, p). \tag{5.13}$$

Then

$$\mathfrak{A}(\mu) = \frac{1}{2} \begin{bmatrix} \varphi_-(\mu)\{I_p + c(\mu)^*\} & \varphi_-(\mu)\{I_p - c(\mu)^*\} \\ \varphi_+(\mu)\{I_p - c(\mu)\} & \varphi_+(\mu)\{I_p + c(\mu)\} \end{bmatrix}, \tag{5.14}$$

where

1. $c \in \mathbb{C}^{p\times p}(\Omega_+)$.
2. $\log\det\{\frac{c(\mu)+c(\mu)^*}{2}\} \in \widetilde{L_1(\Omega_0)}$.
3. *If*

$$\Delta(\mu) = 2\{c(\mu)+c(\mu)^*\}^{-1} \tag{5.15}$$

a.e. on Ω_0, *then*

$$\varphi_-(\mu) = \mathfrak{a}_-(\mu) + \mathfrak{b}_+(\mu) \qquad \textit{and} \qquad \varphi_+(\mu) = \mathfrak{b}_+(\mu) + \mathfrak{a}_+(\mu) \tag{5.16}$$

are the essentially unique solutions of the factorization problems

$$\Delta(\mu) = \varphi_-(\mu)^*\,\varphi_-(\mu) = \varphi_+(\mu)^*\,\varphi_+(\mu) \tag{5.17}$$

such that $\varphi_-^{\#}(\lambda)$ *and* $\varphi_+(\lambda)$ *are outer mvf's and*

$$(\rho_\omega\varphi_-^{\#})^{-1} \in H_2^{p\times p}(\Omega_+) \qquad \textit{and} \qquad (\rho_\omega\varphi_+)^{-1} \in H_2^{p\times p}(\Omega_+) \tag{5.18}$$

for at least one (and hence every) $\omega \in \Omega_+$. *Conversely, if* $\mathfrak{A}(\mu)$ *is given by formula* (5.14) *and* (1)–(3) *hold, then* $\mathfrak{A} \in \mathfrak{M}(p,p)$.

Proof. Let $\mathfrak{A} \in \mathfrak{M}(p,p)$. Then the parametrization formula (5.6) implies the mvf's $\varphi_\pm(\mu)$ that are defined by (5.16) can be reexpressed as

$$\varphi_-(\mu) = \mathfrak{a}_-(\mu)\{I_p - \chi(\mu)^*\} \quad \text{and} \quad \varphi_+(\mu) = \mathfrak{a}_+(\mu)\{I_p - \chi(\mu)\} \tag{5.19}$$

and hence that

$$\mathfrak{A}(\mu) = \begin{bmatrix} \varphi_-(\mu)\{I_p - \chi(\mu)^*\}^{-1} & -\varphi_-(\mu)\{I_p-\chi(\mu)^*\}^{-1}\chi(\mu)^* \\ -\varphi_+(\mu)\{I_p-\chi(\mu)\}^{-1}\chi(\mu) & \varphi_+(\mu)\{I_p-\chi(\mu)\}^{-1} \end{bmatrix} \tag{5.20}$$

a.e. on Ω_0. The parametrization formula (5.14) now emerges easily from (5.20) upon invoking formula (5.9). Moreover, the fact that $c(\lambda)$ satisfies (1) and (2) in the statement of the theorem has already been noted. Property (3) is an easy consequence of formulas (5.3), (5.4), (5.19), (5.11), and the fact that $\mathfrak{a}_+^{-1} \in \mathcal{S}_{out}^{p\times p}$, $(a_-^{\#})^{-1} \in \mathcal{S}_{out}^{p\times p}$ and

$$\{I_p - \chi(\lambda)\}^{-1} = \{I_p + c(\lambda)\}/2 \tag{5.21}$$

is outer. The asserted summability follows from (5.10).

This completes the proof that every $\mathfrak{A} \in \mathfrak{M}(p,p)$ admits a parametrization of the asserted form. Conversely, it is readily checked that every mvf

$\mathfrak{A}(\mu)$ of the form (5.14), whose entries meet properties (1)–(3), belongs to the class $\mathfrak{M}(p, p)$ (that is defined in Section 3). ∎

5.3. *A factorization based on the parametrization in Subsection* 5.2. The next step is to invoke the Riesz–Herglotz integral representation for mvf's $c(\lambda) \in \mathbb{C}^{p\times p}(\Omega_+)$:

$$c(\lambda) = c_a(\lambda) + c_s(\lambda), \tag{5.22}$$

where

$$c_a(\lambda) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{e^{i\theta}+\lambda}{e^{i\theta}-\lambda}\,\Delta(e^{i\theta})^{-1}\,d\theta & \text{if } \Omega_+ = \mathbb{D} \\ \dfrac{1}{\pi i}\displaystyle\int_{-\infty}^{\infty}\left[\dfrac{1}{\mu-\lambda} - \dfrac{\mu}{1+\mu^2}\right]\Delta(\mu)^{-1}\,d\mu & \text{if } \Omega_+ = \mathbb{C}_+ \end{cases} \tag{5.23}$$

and

$$c_s(\lambda) = \begin{cases} i2\gamma + \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{e^{i\theta}+\lambda}{e^{i\theta}-\lambda}\,d\sigma_s(\theta) & \text{if } \Omega_+ = \mathbb{D} \\ i2\gamma - i\beta\lambda + \dfrac{1}{\pi i}\displaystyle\int_{-\infty}^{\infty}\left[\dfrac{1}{\mu-\lambda} - \dfrac{\mu}{1+\mu^2}\right] d\sigma_s(\mu) & \text{if } \Omega_+ = \mathbb{C}_+ \end{cases} \tag{5.24}$$

both belong to $\mathscr{C}^{p\times p}(\Omega_+)$, $\sigma_s(\mu)$ is a nondecreasing bounded $p\times p$ mvf on Ω_0 such that $\sigma_s'(\mu) = 0$ a.e. on Ω_0, $\gamma = \gamma^*$ and $\beta \geqslant 0$. The last formula is valid for a mvf $c_s \in \mathscr{C}^{p\times p}(\Omega_+)$ if and only if

$$c_s(\mu) + c_s(\mu)^* = 0 \qquad \text{a.e. on } \Omega_0. \tag{5.25}$$

THEOREM 5.2. *Let $\mathfrak{A} \in \mathfrak{M}(p, p)$ be parametrized by formula* (5.14) *and let the mvf $c(\lambda) \in \mathscr{C}^{p\times p}$ considered in this formula be expressed in the form* (5.22). *Then the formula for $\mathfrak{A}(\mu)$ can be reexpressed in the following equivalent ways*:

$$\mathfrak{A}(\mu) = \mathfrak{A}_a(\mu) + \tfrac{1}{2}\begin{bmatrix} \varphi_-(\mu)\,c_s(\mu)^* & -\varphi_-(\mu)\,c_s(\mu)^* \\ -\varphi_+(\mu)\,c_s(\mu) & \varphi_+(\mu)\,c_s(\mu) \end{bmatrix}, \tag{5.26}$$

$$\mathfrak{A}(\mu) = \mathfrak{A}_a(\mu) + \tfrac{1}{2}\begin{bmatrix} -\varphi_-(\mu)\,c_s(\mu) & \varphi_-(\mu)\,c_s(\mu) \\ -\varphi_+(\mu)\,c_s(\mu) & \varphi_+(\mu)\,c_s(\mu) \end{bmatrix} \tag{5.27}$$

and

$$\mathfrak{A}(\mu) = \mathfrak{A}_a(\mu)\,\mathfrak{A}_s(\mu), \tag{5.28}$$

where

$$\mathfrak{A}_a(\mu) = \frac{1}{2}\begin{bmatrix} \varphi_-(\mu)\{I_p + c_a(\mu)^*\} & \varphi_-(\mu)\{I_p - c_a(\mu)^*\} \\ \varphi_+(\mu)\{I_p - c_a(\mu)\} & \varphi_+(\mu)\{I_p + c_a(\mu)\} \end{bmatrix} \tag{5.29}$$

and

$$\mathfrak{A}_s(\mu) = \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -c_s(\mu) & c_s(\mu) \\ -c_s(\mu) & c_s(\mu) \end{bmatrix} \tag{5.30}$$

a.e. on Ω_0. Moreover, $\mathfrak{A}_a \in \mathfrak{M}(p, p)$ and $\mathfrak{A}_s \in \mathfrak{M}_S(p, p)$.

Proof. The exhibited formulas are easily obtained by substituting the decomposition (5.22) into formula (5.14). The claim that $\mathfrak{A}_a \in \mathfrak{M}(p, p)$ is also easily verified.

Finally, to show that $\mathfrak{A}_s \in \mathfrak{M}_S(p, p)$, observe first that $\mathfrak{A}_s(\lambda)$ is holomorphic in Ω_+. Therefore we can compute the Potapov–Ginzburg transform $\mathfrak{B}_s(\lambda)$ of $\mathfrak{A}_s(\lambda)$ in Ω_+ to obtain

$$\mathfrak{B}_s(\lambda) = \begin{bmatrix} I_p & d(\lambda) \\ d(\lambda) & I_p \end{bmatrix}\begin{bmatrix} I_p + d(\lambda) & 0 \\ 0 & I_p + d(\lambda) \end{bmatrix}^{-1},$$

where

$$d(\lambda) = c_s(\lambda)/2.$$

Thus, as

$$d(\lambda) + d(\lambda)^* \geqslant 0$$

in Ω_+, we can define

$$E(\lambda) = [\,\{d(\lambda) + d(\lambda)^*\}^{1/2} \quad -\{d(\lambda) + d(\lambda)^*\}^{1/2}]$$

in Ω_+ and verify that

$$I_{2p} - \mathfrak{B}_s(\lambda)^*\,\mathfrak{B}_s(\lambda) = \begin{bmatrix} I_p + d(\lambda) & 0 \\ 0 & I_p + d(\lambda) \end{bmatrix}^{-*} E(\lambda)^*\,E(\lambda)$$

$$\times \begin{bmatrix} I_p + d(\lambda) & 0 \\ 0 & I_p + d(\lambda) \end{bmatrix}^{-1} \geqslant 0$$

in Ω_+. This proves that $\mathfrak{B}_s(\lambda)$ is contractive in Ω_+ and hence that $\mathfrak{A}_s(\lambda)$ is j_p-contractive in Ω_+. Therefore, since $\mathfrak{A}_s(\mu)$ is j_p-unitary a.e. on Ω_0, it follows that $\mathfrak{A}_s(\mu)$ is the boundary value of a j_p-inner function, which we shall continue to refer to as $\mathfrak{A}_s(\lambda)$. Moreover, as

$$\mathfrak{A}_s(\mu)^{-1} = \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} c_s(\mu) & -c_s(\mu) \\ c_s(\mu) & -c_s(\mu) \end{bmatrix}, \tag{5.31}$$

it follows that $\mathfrak{A}_s^{\pm 1} \in \mathcal{N}_+^{m\times m}(\Omega_+)$ and hence that

$$\mathfrak{A}_s \in \mathcal{U}_S(j_p) = \mathfrak{M}_S(p, p),$$

as claimed. ∎

Corollary 5.3. *The class of constant j_p-unitary matrices is parametrized by the formulas*

$$\mathfrak{A} = \mathfrak{A}_a \mathfrak{A}_s \tag{5.32}$$

$$\mathfrak{A}_a = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} (\delta^{1/2}+\delta^{-1/2})/2 & (\delta^{1/2}-\delta^{-1/2})/2 \\ (\delta^{1/2}-\delta^{-1/2})/2 & (\delta^{1/2}+\delta^{1/2})/2 \end{bmatrix} \tag{5.33}$$

and

$$\mathfrak{A}_s = \begin{bmatrix} I_p - i\gamma & i\gamma \\ -i\gamma & I_p + i\gamma \end{bmatrix}, \tag{5.34}$$

where u, v, δ and γ are arbitrary $p\times p$ matrices such that

$$u^*u = v^*v = I_p, \qquad \delta > 0 \qquad \text{and} \qquad \gamma^* = \gamma. \tag{5.35}$$

Proof. The class of mvf's $\mathfrak{A} \in \mathfrak{M}(p, q)$ which are constant on Ω_0 coincides with the class of constant j_{pq}-unitary matrices. Therefore the sought for parametrization can be obtained by specializing formulas (5.29) and (5.30) to the case when $\mathfrak{A}(\mu)$ is constant. In particular, this assumption implies that $\varphi_-(\mu)$, $\varphi_+(\mu)$, $\Delta(\mu)$, $c_a(\mu)$ and $c_s(\mu)$ are all constant. Upon setting

$$\Delta(\mu) = \delta > 0,$$

it follows that

$$\varphi_-(\mu) = u\delta^{1/2}, \qquad \varphi_+(\mu) = v\delta^{1/2}$$
$$c_a(\mu) = \delta^{-1} \qquad \text{and} \qquad c_s(\mu) = i2\gamma,$$

where u, v, δ and γ are subject to (5.35). The rest is immediate from formulas (5.28)–(5.30). ∎

5.4. *Regular-singular factorization of* $\mathfrak{A} \in \mathfrak{M}(p, p)$. If $\mathfrak{A} \in \mathfrak{M}_R(p, p)$, then in the representation

$$\mathfrak{A}(\mu) = \mathfrak{A}_a(\mu)\,\mathfrak{A}_s(\mu)$$

that was established above, the facto $\mathfrak{A}_s(\mu)$ is a constant j_p-unitary matrix of the form (5.34). The converse is not true because in general there is no guarantee that $\mathfrak{A}_a \in \mathfrak{M}_R(p, p)$. Our next objective is to obtain necessary and sufficient conditions for this to happen. We shall need the notion of a mvf with zero index.

Let $g(\mu)$ be a $p \times p$ measurable mvf that is unitary a.e. on Ω_0 and admits a factorization of the form

$$g(\mu) = \psi_-(\mu)\,\psi_+(\mu)^{-1}, \tag{5.36}$$

where

$$\rho_\omega^{-1}(\psi_-^\#)^{-1} \in H_2^{p\times p}(\Omega_+) \qquad \text{and} \qquad \rho_\omega^{-1}\psi_+^{-1} \in H_2^{p\times p}(\Omega_+) \tag{5.37}$$

for at least one (and hence every) $\omega \in \Omega_+$. Then we shall say that $\operatorname{index}\{g\} = 0$ if for any other pair of mvf's $\widetilde{\psi}_-$, $\widetilde{\psi}_+$ with the properties (5.36) and (5.37) the equalities

$$\widetilde{\psi}_-(\mu) = \psi_-(\mu)\,k \qquad \text{and} \qquad \psi_+(\mu) = \widetilde{\psi}_+(\mu)\,k \tag{5.38}$$

hold a.e. on Ω_0 for some invertible constant $p \times p$ matrix k.

We remark that this automatically forces $\psi_-^\#(\lambda)$ and $\psi_+(\lambda)$ to be outer mvf's.

THEOREM 5.4. *Let* $\mathfrak{A} \in \mathfrak{M}(p, p)$ *be expressed in the form*

$$\mathfrak{A}(\mu) = \mathfrak{A}_a(\mu)\,\mathfrak{A}_s(\mu) \tag{5.39}$$

considered in the preceding theorem. Then

$$\mathfrak{A}_a \in \mathfrak{M}_R(p, p) \Leftrightarrow \operatorname{index}\{T_{\mathfrak{A}}[I_p]\} = 0 \tag{5.40}$$

Proof. The implication

$$\mathfrak{A} \in \mathfrak{M}_R(p, p) \Rightarrow \operatorname{index}\{T_{\mathfrak{A}}[\mathscr{E}]\} = 0 \qquad \text{for every unitary} \quad \mathscr{E} \in \mathbb{C}^{p\times p}$$

follows from [AAK1] if $p = 1$ and from [Ad] if $p > 1$. This serves to justify the claim that

$$\mathfrak{A}_a \in \mathfrak{M}_R(p, p) \Rightarrow \operatorname{index}\{T_{\mathfrak{A}}[I_p]\} = 0,$$

since

$$\{T_{\mathfrak{A}}[I_p]\} = \{T_{\mathfrak{A}_a}[I_p]\}. \tag{5.41}$$

Suppose next that

$$\text{index}\{T_{\mathfrak{A}}[I_p]\} = 0$$

and let

$$g = T_{\mathfrak{A}}[I_p] = T_{\mathfrak{A}_a}[I_p].$$

By a general result due to [Ar6] the fact that $g \in T_{\mathfrak{A}}[\mathscr{S}^{p\times p}]$ guarantees that the NP(g, Ω_0) is completely indeterminate. Therefore, there exists an essentially unique mvf $\mathfrak{A}_1 \in \mathfrak{M}_R(p, p)$ such that

$$T_{\mathfrak{A}_1}[\mathscr{S}^{p\times p}] = \mathscr{F}(g) \tag{5.42}$$

and

$$\mathfrak{A}_a(\mu) = \mathfrak{A}_1(\mu)\, \mathfrak{A}_2(\mu) \tag{5.43}$$

a.e. on Ω_0, where $\mathfrak{A}_2 \in \mathfrak{M}_S(p, p)$. Thus it remains to show that $\mathfrak{A}_2(\mu)$ is constant.

In view of (5.42),

$$g = T_{\mathfrak{A}_1}[\mathscr{E}_1]$$

for some $\mathscr{E}_1 \in \mathscr{S}^{p\times p}(\Omega_+)$. Moreover, since $g(\mu)$ is unitary a.e. on Ω_0, $\mathscr{E}_1 \in \mathscr{S}_{in}^{p\times p}(\Omega_+)$. Consequently, upon writing

$$\mathfrak{A}_1(\mu) = \begin{bmatrix} \mathfrak{a}_-^{(1)}(\mu) & \mathfrak{b}_-^{(1)}(\mu) \\ \mathfrak{b}_+^{(1)}(\mu) & \mathfrak{a}_+^{(1)}(\mu) \end{bmatrix} \quad \text{and} \quad \chi_1(\mu) = -\{\mathfrak{a}_+^{(1)}(\mu)\}^{-1}\, \mathfrak{b}_+^{(1)}(\mu) \tag{5.44}$$

we obtain the identity

$$(\mathfrak{a}_-^{(1)} + \mathfrak{b}_-^{(1)} \mathscr{E}_1^*)\, \mathscr{E}_1 (\mathfrak{a}_+^{(1)} + \mathfrak{b}_+^{(1)} \mathscr{E}_1)^{-1} = \varphi_- \varphi_+^{-1}$$

a.e. on Ω_0. But this is of the form

$$\psi_- \psi_+^{-1} = \varphi_- \varphi_+^{-1}$$

with

$$\psi_-(\mu) = \mathfrak{a}_-^{(1)}(\mu) + \mathfrak{b}_-^{(1)}(\mu)\, \mathscr{E}_1(\mu)^*$$

and

$$\psi_+(\mu)^{-1} = \mathscr{E}_1(\mu)\{\mathfrak{a}_+^{(1)}(\mu) + \mathfrak{b}_+^{(1)}(\mu)\,\mathscr{E}_1(\mu)\}^{-1}.$$

Therefore, since $\mathfrak{A}_1 \in \mathfrak{M}_R(p, p)$, it follows from the analysis in [Ar6] that $\psi_+(\mu)$ and $\psi_-(\mu)$ meet condition (5.37) and thus the assumption that index$\{g\} = 0$ guarantees the existence of an invertible constant k such that

$$\psi_+(\mu) = \varphi_+(\mu)\,k \qquad \text{and} \qquad \psi_-(\mu) = \varphi_-(\mu)\,k$$

a.e. on Ω_0. But the first of these relations implies $\psi_+(\lambda)^{-1}$ is outer and therefore, since

$$\{\mathfrak{a}_+^{(1)}(\lambda) + \mathfrak{b}_+^{(1)}(\lambda)\,\mathscr{E}_1(\lambda)\}^{-1} = \{I_p - \chi_1(\lambda)\,\mathscr{E}_1(\lambda)\}^{-1}\,\mathfrak{a}_+^{(1)}(\lambda)^{-1}$$

is outer, $\mathscr{E}_1(\lambda)$ is a unitary constant. Thus, upon setting

$$\mathfrak{A}_3(\mu) = \mathfrak{A}_1(\mu)\begin{bmatrix} \mathscr{E}_1 & 0 \\ 0 & I_p \end{bmatrix},$$

we obtain the formula

$$T_{\mathfrak{A}_3}[I_p] = T_{\mathfrak{A}_1}[\mathscr{E}_1] = T_{\mathfrak{A}_a}[I_p].$$

Next let $\widetilde{\mathfrak{A}}(\mu) = \mathfrak{A}_3(\mu)\,U$, where

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

is a constant j_p-unitary matrix such that

$$u_{11} + u_{12} = u_{21} + u_{22} = (k\mathscr{E}_1)^{-1}.$$

The parametrization formulas (5.32)–(5.34) guarantee that this is possible. Then it is readily checked that

$$\widetilde{\mathfrak{A}}(\mu)\begin{bmatrix} I_p \\ I_p \end{bmatrix} = \begin{bmatrix} \varphi_-(\mu) \\ \varphi_+(\mu) \end{bmatrix} = \mathfrak{A}_a(\mu)\begin{bmatrix} I_p \\ I_p \end{bmatrix}$$

a.e. on Ω_0. Therefore, in the factorization

$$\widetilde{\mathfrak{A}}(\mu) = \widetilde{\mathfrak{A}}_a(\mu)\,\widetilde{\mathfrak{A}}_s(\mu)$$

based on formulas (5.28)–(5.30),

$$\widetilde{\mathfrak{A}}_a(\mu) = \widetilde{\mathfrak{A}}_a(\mu)$$

a.e. on Ω_0 and $\widetilde{\mathfrak{A}}_s(\mu)$ is constant, since $\widetilde{\mathfrak{A}} \in \mathfrak{M}_R(p, p)$. Thus $\mathfrak{A}_a \in \mathfrak{M}_R(p, p)$ as claimed. ▮

THEOREM 5.5. *Let $\mathfrak{A} \in \mathfrak{M}(p, p)$ be expressed in the form*

$$\mathfrak{A}(\mu) = \mathfrak{A}_a(\mu)\, \mathfrak{A}_s(\mu)$$

considered in Theorem 5.2 *and let*

$$c(\lambda) = c_a(\lambda) + c_s(\lambda)$$

be defined by formulas (5.17), (5.23) *and* (5.24). *Then the following statements are equivalent*:

1. $\mathfrak{A} \in \mathfrak{M}_R(p, p)$.
2. $\mathfrak{A}_a \in \mathfrak{M}_R(p, p)$ *and* $\mathfrak{A}_s(\mu)$ *is constant.*
3. $\operatorname{index}\{T_{\mathfrak{A}}[I_p]\} = 0$ *and* $\mathfrak{A}_s(\mu)$ *is constant.*
4. $\operatorname{index}\{T_{\mathfrak{A}}[I_p]\} = 0$ *and* $c_s(\lambda)$ *is constant.*
5. $\operatorname{index}\{T_{\mathfrak{A}}[\mathscr{E}]\} = 0$ *for every constant unitary* $p \times p$ *matrix* $\mathscr{E}$ *and* $\mathfrak{A}_s(\mu)$ *is constant.*

Proof. The implications $(1) \Leftrightarrow (2)$, $(3) \Leftrightarrow (4)$ and $(5) \Rightarrow (3)$ are selfevident; $(1) \Rightarrow (5)$ is established in [AAK1] for $p = 1$ and in [Ad] for $p > 1$. Finally, $(3) \Rightarrow (2)$, by the preceding theorem. ▮

LEMMA 5.6. *Let $\Delta(\mu)$ be a $p \times p$ measurable mvf on Ω_0 such that $\Delta(\mu) > 0$ a.e. and $\Delta^{\pm 1} \in \widetilde{L_1^{p \times p}}(\Omega_0)$. Suppose further that*

$$\Delta(\mu) = \varphi_-(\mu)^*\, \varphi_-(\mu) = \varphi_+(\mu)^*\, \varphi_+(\mu) \text{ a.e. on } \quad \Omega_0, \tag{5.45}$$

where

$$\begin{gathered} \rho_\omega^{-1}\varphi_+,\ \rho_\omega^{-1}\varphi_+^{-1},\ \rho_\omega^{-1}\varphi_-^{\#} \text{ and } \rho_\omega^{-1}(\varphi_-^{\#})^{-1} \\ \text{are outer functions in } H_2^{p \times p}(\Omega_+) \end{gathered} \tag{5.46}$$

for at least one (*and hence every*) $\omega \in \Omega_+$. *Then*

$$\operatorname{index}\{\varphi_-(\mu)\, \varphi_+(\mu)^{-1}\} = 0.$$

Proof. Suppose that

$$\varphi_-(\mu)\,\varphi_+(\mu)^{-1} = \psi_-(\mu)\,\psi_+(\mu)^{-1},$$

where $\psi_-(\mu)$ and $\psi_+(\mu)$ are $p \times p$ mvf's that meet conditions (5.45) and (5.46). Then

$$\psi_-(\mu)^{-1}\,\varphi_-(\mu) = \psi_+(\mu)^{-1}\,\varphi_+(\mu)$$

a.e. on Ω_0. Therefore, in view of assumption (5.46),

$$\psi_+^{-1}\varphi_+ \in H_1^{p\times p}(\mathbb{D}) \cap \overline{H_1^{p\times p}(\mathbb{D})}$$

when $\Omega_+ = \mathbb{D}$, and hence is constant. The case $\Omega_+ = \mathbb{C}_+$ follows by invoking the conformal mapping ψ in (3.5) of $\mathbb{D}$ onto $\mathbb{C}_+$ and exploiting the fact that if $\omega \in \mathbb{C}_+$, then

$$\frac{f}{\rho_\omega} \in H_2^{p\times p}(\mathbb{C}_+) \Leftrightarrow f \circ \psi \in H_2^{p\times p}(\mathbb{D}). \quad \blacksquare$$

THEOREM 5.7. *Let $\mathfrak{A} \in \mathfrak{M}(p, p)$, let*

$$\Delta(\mu) = \varphi_-(\mu)^*\,\varphi_-(\mu) = \varphi_+(\mu)^*\,\varphi_+(\mu)$$

a.e. on $\underset{\sim}{\Omega_0}$, where $\varphi_\pm(\mu)$ are defined by formula (5.16) *and suppose that $\Delta^{\pm 1} \in L_1^{p\times p}(\Omega_0)$. Then the factorization*

$$\mathfrak{A}(\mu) = \mathfrak{A}_a(\mu)\,\mathfrak{A}_s(\mu)$$

based on formulas (5.29) *and* (5.30) *is a regular-singular factorization, i.e.,*

$$\mathfrak{A}_a \in \mathfrak{M}_R(p, p) \qquad \textit{and} \qquad \mathfrak{A}_s \in \mathfrak{M}_s(p, p).$$

Proof. The preceding lemma guarantees that

$$\text{index}\{T_{\mathfrak{A}_a}[I_p]\} = 0.$$

Thus $\mathfrak{A}_a \in \mathfrak{M}_R(p, p)$, by Theorem 5.4. This completes the proof, since $\mathfrak{A}_s(\mu)$ is always singular. $\blacksquare$

5.5. *More on the class $\mathfrak{M}_{sR}(p, p)$.*

THEOREM 5.8. *Let $\Delta(\mu)$ be a $p \times p$ measurable mvf on Ω_0 that is positive definite a.e. and meets the matricial Muckenhoupt condition* (4.2). *Then the* *mvf* $\mathfrak{A}_a(\mu)$ *that is determined by* $\Delta(\mu)$ *and the solutions* $\varphi_{\pm}(\mu)$ *of the factorization problem* (5.17) *and* (5.18) *via formulas* (5.23) *and* (5.29) *is strongly regular. Moreover, the family of mvf's of the form*

$$\mathfrak{A}(\mu)=\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \mathfrak{A}_a(\mu) \begin{bmatrix} I_p - i\gamma & i\gamma \\ -i\gamma & I_p + i\gamma \end{bmatrix} \tag{5.47}$$

with $u^*u=v^*v=I_p$ *and* $\gamma=\gamma^*$ *is a full parametrization of the set of* $\mathfrak{A}\in\mathfrak{M}_{sR}(p,p)$ *of the form* (5.13) *with*

$$\begin{aligned}\{\mathfrak{a}_-(\mu)+\mathfrak{b}_-(\mu)\}^*\{\mathfrak{a}_-(\mu)+\mathfrak{b}_-(\mu)\} &= \{\mathfrak{a}_+(\mu)+\mathfrak{b}_+(\mu)\}^*\{\mathfrak{a}_+(\mu)+\mathfrak{b}_+(\mu)\} \\ &= \Delta(\mu) \end{aligned} \tag{5.48}$$

a.e. on Ω_0.

Proof. If $\Delta(\mu)$ meets the matricial Muckenhoupt condition (4.2), then

$$\Delta^{\pm 1}\in \widetilde{L_1^{p\times p}}(\Omega_0).$$

Thus,

$$\operatorname{index}\{T_{\mathfrak{A}}[I_p]\}=\operatorname{index}\{\varphi_-\varphi_+^{-1}\}=0,$$

by Lemma 5.6. Consequently, $\mathfrak{A}_a\in\mathfrak{M}_R(p,p)$, by Theorem 5.4. Moreover, by Theorem 4.2 and Lemma 4.4, the Hankel operator Γ_g based on $g=\varphi_-\varphi_+^{-1}$ is strictly contractive. Therefore, there exists an $f\in\mathscr{F}(g)$ with $\|f\|_\infty<1$. However, since $\mathfrak{A}_a\in\mathfrak{M}_R(p,p)$,

$$T_{\mathfrak{A}_a}[\mathscr{S}^{p\times p}]=\mathscr{F}(g)$$

and hence there exists a mvf $\mathscr{E}\in\mathscr{S}^{p\times p}(\Omega_+)$ such that

$$\|T_{\mathfrak{A}_a}[\mathscr{E}]\|_\infty=\|f\|_\infty<1.$$

This proves that $\mathfrak{A}_a\in\mathfrak{M}_{sR}(p,p)$.

Finally, if $\mathfrak{A}\in\mathfrak{M}_{sR}(p,p)$ is any strongly regular mvf of the form (5.13) for which (5.48) holds, then

$$\mathfrak{A}(\mu)=\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \mathfrak{A}_a(\mu)\,\mathfrak{A}_s(\mu)$$

a.e. on Ω_0, where u and v are unitary and $\mathfrak{A}_s(\mu)$ is constant. In view of formula (5.34), this agrees with the form asserted in the theorem. ∎

5.6. *An example.* Let $\Omega_+ = \mathbb{D}$ and let

$$\Delta_\alpha(\mu) = |1-\mu|^{-2\alpha} = \left|2\sin\left(\frac{\theta}{2}\right)\right|^{-2\alpha} \qquad \text{for} \quad \mu \in \mathbb{T} \quad (\mu = e^{i\theta}),$$

$$\varphi_+^\alpha(\lambda) = (1-\lambda)^{-\alpha} \qquad \text{for} \quad \lambda \in \mathbb{C}\backslash[1, \infty)$$

and

$$\varphi_-^\alpha(\lambda) = (1-1/\lambda)^{-\alpha} \qquad \text{for} \quad \lambda \in \mathbb{C}\backslash[0, 1],$$

where $\varphi_+^\alpha(\lambda) > 0$ for $0 < \lambda < 1$ and $\varphi_-^\alpha(\lambda) > 0$ for $\lambda > 1$. From now on, in order to have $\Delta_\alpha(\mu)^{-1} \in L_1(\mathbb{T})$, we shall assume that $\alpha > -\frac{1}{2}$. Then $\varphi_+^\alpha(\lambda)^{-1}$ is an outer Hardy function in $H_2(\mathbb{D})$ such that

$$(\varphi_-^\alpha)^\#(\lambda) = \varphi_+^\alpha(\lambda),$$

$$\varphi_-^\alpha(\mu)^* \varphi_-^\alpha(\mu) = \varphi_+^\alpha(\mu)^* \varphi_+^\alpha(\mu) = \Delta_\alpha(\mu)$$

and

$$g_\alpha(\mu) = \varphi_-^\alpha(\mu)\, \varphi_+^\alpha(\mu)^{-1} = (-1/\mu)^{-\alpha} = e^{i(\theta-\pi)\alpha}$$

for a.e. $\mu \in \mathbb{T}$. Therefore, by the evaluations cited in Section 5 of [AAK1] the norm of the Hankel operator Γ_{g_α} is given by the formula

$$\|\Gamma_{g_\alpha}\| = |\sin \pi\alpha| \qquad (\alpha > -\tfrac{1}{2}, \alpha \text{ not an integer}).$$

Now let $c_a^\alpha(\lambda)$ denote the Carathéodory function that is defined by formula (5.23) with $\Delta(e^{i\theta}) = \Delta_\alpha(e^{i\theta})$ and let $\mathfrak{A}_a^\alpha(\mu)$ denote the 2×2 mvf that is defined in terms of $\varphi_-(\mu) = \varphi_-^\alpha(\mu)$, $\varphi_+(\mu) = \varphi_+^\alpha(\mu)$ and $c_a(\mu) = c_a^\alpha(\mu)$ by formula (5.29).

Theorem 5.9. *Let $\mathfrak{A}_a^\alpha(\mu)$ be defined as above. Then the following conclusions prevail:*

1. *If $-\frac{1}{2} < \alpha < \frac{1}{2}$, then $\Delta_\alpha(\mu)$ is a Muckenhoupt weight and $\mathfrak{A}_a^\alpha \in \mathfrak{M}_{sR}(1, 1)$.*
2. *If $\alpha = \frac{1}{2}$, then*

$$\mathfrak{A}_a^\alpha \notin \mathfrak{M}_{sR}(1, 1), \qquad \textit{but}$$

$$\text{index}\{g_\alpha\} = 0 \qquad \textit{and} \qquad \mathfrak{A}_a^\alpha \in \mathfrak{M}_R(1, 1).$$

3. *If $\alpha > \frac{1}{2}$, then*

$$\text{index}\{g_\alpha\} \neq 0 \qquad \textit{and} \qquad \mathfrak{A}_a^\alpha \notin \mathfrak{M}_R(1, 1).$$

Proof. If $0<|\alpha|<\frac{1}{2}$, then $\Delta_\alpha \in L_1(\mathbb{T})$ and $\|\Gamma_{g_\alpha}\|<1$. Therefore, by Theorem 4.2 and Lemma 4.4, $\Delta_\alpha(\mu)$ satisfies the Muckenhoupt condition (4.2). On the other hand, if $\alpha=0$, then $\Delta_\alpha(\mu)=1$, which also clearly satisfies the condition (4.2). Therefore $\mathfrak{A}_a^\alpha \in \mathfrak{M}_{sR}(1,1)$, by Theorem 5.8, for $|\alpha|<\frac{1}{2}$.

If $\alpha \geqslant \frac{1}{2}$, then $\Delta_\alpha \notin L_1(\mathbb{T})$ and therefore cannot satisfy the Muckenhoupt condition (4.2). However, if $\alpha=\frac{1}{2}$, then it follows from Lemma 5.1 of [AAK1] that $\text{index}\{g_\alpha\}=0$. Therefore $\mathfrak{A}_a^\alpha \in \mathfrak{M}_R(1,1)$ for $\alpha=\frac{1}{2}$, by Theorem 5.4.

Finally, if $\alpha>\frac{1}{2}$, then

$$
\begin{aligned}
g_\alpha(\mu) &= \frac{(1-\bar{\mu})^{-\alpha}}{(1-\mu)^{-\alpha}} \\
&= \left(\frac{1-\bar{\mu}}{1-\mu}\right)^{-1} \frac{\varphi_-^{\alpha-1}(\mu)}{\varphi_+^{\alpha-1}(\mu)} \\
&= -\mu \frac{\varphi_-^{\alpha-1}(\mu)}{\varphi_+^{\alpha-1}(\mu)}.
\end{aligned}
$$

This exhibits a second factorization of $g_\alpha(\mu)$ of the form $\psi_-(\mu)\,\psi_+(\mu)^{-1}$ with

$$
\psi_-(\lambda) = -\lambda\varphi_-^{\alpha-1}(\lambda) \qquad \text{and} \qquad \psi_+(\lambda)=\varphi_+^{\alpha-1}(\lambda).
$$

Therefore, since $(\psi_-^{\#})^{-1} \in H_2(\mathbb{D})$ and $\psi_+^{-1} \in H_2(\mathbb{D})$ but there does not exist a constant k such that $\psi_-(\lambda)=k\varphi_-^\alpha(\lambda)$ and $\psi_+(\lambda)=k\varphi_+^\alpha(\lambda)$, $\text{index}\{g_\alpha\}\neq 0$. The rest is immediate from Theorem 5.4. ▮

THEOREM 5.10. *The inclusion*

$$
L_\infty^{2\times 2}(\mathbb{T}) \cap \mathfrak{M}(1,1) \subset \mathfrak{M}_{sR}(1,1)
$$

is proper.

Proof. Let $\mathfrak{A}_a^\alpha(\mu)$ be the mvf that was considered in the preceding theorem. Then $\mathfrak{A}_a^\alpha \in \mathfrak{M}_{sR}(1,1)$ when $|\alpha|<1$. However, if $0<\alpha<\frac{1}{2}$, then the mvf's $\varphi_+^\alpha(\mu)$ and $\varphi_-^\alpha(\mu)$ are both unbounded. Therefore $\mathfrak{A}_a^\alpha \notin L_\infty^{2\times 2}(\mathbb{T})$. Moreover, if $-\frac{1}{2}<\alpha<0$, then

$$
\Re c_a^\alpha(\mu) = \Delta_\alpha(\mu)^{-1} = |1-\mu|^{2\alpha}
$$

is unbounded. Therefore $c_a(\mu)$ is unbounded and $\mathfrak{A}_a^\alpha \notin L_\infty^{2\times 2}(\mathbb{T})$ for this range of α also. ▮

Remark 5.11. We remark that

$$\| T_{\mathfrak{A}_a^\alpha}[\mathscr{E}] \|_\infty = 1$$

for the mvf $\mathfrak{A}_a^\alpha(\mu)$, $0 < |\alpha| < \frac{1}{2}$, considered in the previous two theorems for every number $\mathscr{E}$ with $|\mathscr{E}| < 1$. In particular, the maximum entropy solution of the associated Nehari problem has L_∞ norm equal to one even though the Hankel operator $\|\Gamma_{g_\alpha}\|$ is strictly contractive:

$$\|\Gamma_{g_\alpha}\| < 1.$$

This was first noted by Bakonyi [Bak]. The conclusion is a consequence of the fact that if $\mathfrak{A} \in \mathfrak{M}(p, q)$, then $\|T_{\mathfrak{A}}[\varepsilon]\|_\infty < 1$ for at least one strictly contractive constant $\mathscr{E} \in \mathscr{S}^{p\times q}(\Omega_+)$ if and only if $\mathfrak{A} \in L_\infty^{m\times m}(\Omega_0)$. Moreover, in this case, $\|T_{\mathfrak{A}}[\mathscr{E}]\|_\infty < 1$ for every strictly contractive constant $\mathscr{E} \in \mathscr{S}^{p\times q}(\Omega_+)$.

6. PARAMETRIZATION AND CHARACTERIZATIONS OF J-INNER MVF'S AND THE SUBCLASSES $\mathscr{U}_{rR}(J)$ AND $\mathscr{U}_{sR}(J)$

In this section we shall study the subclasses $\mathscr{U}_{rR}(J)$ and $\mathscr{U}_{sR}(J)$ of $\mathscr{U}(J)$ for $J = j_{pq}$ and $J = j_p$, where j_p is short for j_{pp}. We begin with a parametrization of mvf's in the class $\mathscr{U}(j_{pq})$, followed by a second parametrization when $q = p$.

6.1. *First parametrization of j_{pq}-inner mvf's.* Every mvf $W \in \mathscr{U}(j_{pq})$ admits a representation of the form

$$W(\mu) = \begin{bmatrix} b_1(\mu) & 0 \\ 0 & b_2(\mu)^{-1} \end{bmatrix} \mathfrak{A}(\mu) \tag{6.1}$$

a.e. on Ω_0, where $b_1 \in \mathscr{S}_{in}^{p\times p}(\Omega_+)$, $b_2 \in \mathscr{S}_{in}^{q\times q}(\Omega_+)$, $\mathfrak{A} \in \mathfrak{M}(p, q)$ and

$$b_1 T_{\mathfrak{A}}[0_{p\times q}]\, b_2 \in \mathscr{S}^{p\times q}(\Omega_+). \tag{6.2}$$

It is known [Ar6, 7] that $W(\mu)$ and $\mathfrak{A}(\mu)$ have essentially unique factorizations:

$$W(\mu) = W_1(\mu)\, W_2(\mu), \qquad \text{where} \quad W_1 \in \mathscr{U}_{rR}(j_{pq}) \quad \text{and} \quad W_2 \in \mathscr{U}_S(j_{pq}). \tag{6.3}$$

$$\mathfrak{A}(\mu) = \mathfrak{A}_1(\mu)\, \mathfrak{A}_2(\mu), \qquad \text{where} \quad \mathfrak{A}_1 \in \mathfrak{M}_R(p, q) \quad \text{and} \quad \mathfrak{A}_2 \in \mathfrak{M}_S(p, q). \tag{6.4}$$

Moreover,

$$W_1(\mu)=\begin{bmatrix} b_1(\mu) & 0 \\ 0 & b_2(\mu)^{-1} \end{bmatrix} \mathfrak{A}_1(\mu) \qquad \text{and} \qquad W_2(\mu)=\mathfrak{A}_2(\mu) \tag{6.5}$$

a.e. on Ω_0.

The parametrization (5.6) of $\mathfrak{A}(\mu)$ in terms of $\chi \in \mathscr{S}^{q\times p}(\Omega_+)$ yields a parametrization of $W(\mu)$ in terms of the three mvf's $b_1(\lambda)$, $b_2(\lambda)$ and $\chi(\lambda)$. In view of the identity

$$\chi(\lambda)=-\mathfrak{a}_+(\lambda)^{-1}\,\mathfrak{b}_+(\lambda)=-w_{22}(\lambda)^{-1}\,w_{21}(\lambda) \tag{6.6}$$

between the indicated blocks of $\mathfrak{A}(\mu)$ and

$$W(\lambda)=\begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}, \tag{6.7}$$

it follows that

$$I_q-\chi(\mu)\,\chi(\mu)^*>0 \text{ a.e. on } \Omega_0. \tag{6.8}$$

Moreover, $\chi(\lambda)$ has a meromorphic pseudocontinuation into Ω_- with bounded Nevanlinna characteristic in Ω_-. Thus, we see that the mvf $\chi(\lambda)$ that arises from any mvf $W\in\mathscr{U}(j_{pq})$ via formula (6.6) has the following two properties:

$$\chi\in\Pi\cap\mathscr{S}^{q\times p}(\Omega_+) \qquad \text{and} \qquad I_q-\chi(\mu)\,\chi(\mu)^*>0 \text{ a.e. on } \Omega_0. \tag{6.9}$$

Now, conversely, let us start with a mvf $\chi(\lambda)$ that enjoys the two properties listed in (6.9) and show that there exists at least one mvf $W\in\mathscr{U}(j_{pq})$ with block decomposition (6.7) (that is conformable with j_{pq}) such that $\chi=-w_{22}^{-1}w_{21}$. Moreover, we shall give a complete description of the set of all such W.

The conditions (6.9) ensure that $I_q-\chi(\mu)\,\chi(\mu)^*$ is the boundary value of a mvf in $\mathscr{N}^{q\times q}(\Omega_+)$ and hence that

$$\log\det\{I_q-\chi(\mu)\,\chi(\mu)^*\}\in\tilde{L}_1(\Omega_0). \tag{6.10}$$

Therefore, the factorization problems

$$\{I_p-\chi(\mu)^*\,\chi(\mu)\}^{-1}=\mathfrak{a}_-(\mu)^*\,\mathfrak{a}_-(\mu), \tag{6.11}$$

$$\{I_q-\chi(\mu)\,\chi(\mu)^*\}^{-1}=\mathfrak{a}_+(\mu)^*\,\mathfrak{a}_+(\mu), \tag{6.12}$$

have essentially unique solutions $\mathfrak{a}_{\pm}(\mu)$ such that

$$(\mathfrak{a}_{-}^{\pm})^{-1} \in \mathscr{S}_{out}^{p\times p}(\Omega_+) \qquad \text{and} \qquad \mathfrak{a}_{+}^{-1} \in \mathscr{S}_{out}^{q\times q}(\Omega_+). \tag{6.13}$$

Then condition (6.9) guarantees that $\mathfrak{a}_{-}^{\#}(\lambda)$ and $\mathfrak{a}_{+}(\lambda)$ both have meromorphic pseudocontinuations with bounded Nevanlinna characteristics in Ω_-. Consequently, we can consider $\mathfrak{a}_{-}(\lambda)$ and $\mathfrak{a}_{+}(\lambda)$ as meromorphic mvf's in $\mathbb{C}\backslash\Omega_0$ with bounded Nevanlinna characteristic in both Ω_+ and Ω_-. Thus, the mvf's

$$\mathfrak{b}_{+}(\lambda) = -\mathfrak{a}_{+}(\lambda)\,\chi(\lambda), \qquad \mathfrak{b}_{-}(\lambda) = -\mathfrak{a}_{-}(\lambda)\,\chi^{\#}(\lambda) \tag{6.14}$$

and

$$f_0(\lambda) = \mathfrak{b}_{-}(\lambda)\,\mathfrak{a}_{+}(\lambda)^{-1} \tag{6.15}$$

enjoy the same properties. Therefore, there exists a pair $\{b_1, b_2\}$ of inner mvf's, $b_1 \in \mathscr{S}_{in}^{p\times p}(\Omega_+)$ and $b_2 \in \mathscr{S}_{in}^{q\times q}(\Omega_+)$, such that

$$s_{12}(\lambda) = b_1(\lambda)\, f_0(\lambda)\, b_2(\lambda) \in \mathscr{N}_{+}^{p\times q}(\Omega_+). \tag{6.16}$$

Such a pair is called a denominator of $f_0(\lambda)$ in [Ar2].

Since $f_0(\mu)$ is contractive a.e. on Ω_0, the Smirnov maximum principle guarantees that $s_{12} \in \mathscr{S}^{p\times q}(\Omega_+)$. In fact,

$$s_{12} \in \Pi \cap \mathscr{S}^{p\times q}(\Omega_+). \tag{6.17}$$

Now define a meromorphic mvf $W(\lambda)$ in Ω_+ by the formula

$$W(\lambda) = \begin{bmatrix} b_1(\lambda) & 0 \\ 0 & b_2(\lambda)^{-1} \end{bmatrix} \begin{bmatrix} \mathfrak{a}_{-}(\lambda) & \mathfrak{b}_{-}(\lambda) \\ \mathfrak{b}_{+}(\lambda) & \mathfrak{a}_{+}(\lambda) \end{bmatrix}. \tag{6.18}$$

The Potapov–Ginzburg transform of $W(\lambda)$,

$$\widetilde{S}(\lambda) = \begin{bmatrix} b_1(\lambda)\,\mathfrak{a}_{-}^{\#}(\lambda)^{-1} & s_{12}(\lambda) \\ \chi(\lambda) & \mathfrak{a}_{+}(\lambda)^{-1}\, b_2(\lambda) \end{bmatrix},$$

belongs to $H_{\infty}^{m\times m}(\Omega_+)$ and is unitary a.e. on Ω_0. Therefore, it belongs to the class $\mathscr{S}_{in}^{m\times m}(\Omega_+)$. Thus, $W \in \mathscr{U}(j_{pq})$. In view of the relations (6.14), the formula for $W(\lambda)$ can be reexpressed as

$$W(\lambda) = \begin{bmatrix} b_1(\lambda) & 0 \\ 0 & b_2(\lambda)^{-1} \end{bmatrix} \begin{bmatrix} \mathfrak{a}_{-}(\lambda) & -\mathfrak{a}_{-}(\lambda)\,\chi^{\#}(\lambda) \\ -\mathfrak{a}_{+}(\lambda)\,\chi(\lambda) & \mathfrak{a}_{+}(\lambda) \end{bmatrix}. \tag{6.19}$$

The preceding analysis serves to solve the following inverse problem: Given a mvf $\chi(\lambda)$ that meets the conditions (6.9), find a mvf $W \in \mathscr{U}(j_{pq})$ such that $\chi(\lambda) = -w_{22}(\lambda)^{-1} w_{21}(\lambda)$. This problem is equivalent to the problem of the Darlington representation wherein the given data is a mvf $s \in \mathscr{S}^{p\times q}(\Omega_+)$ and the question is: When does here exist a mvf $W \in \mathscr{U}(j_{pq})$ such that $T_W[0] = s$? The answer is: If and only if

$$s \in \Pi \cap \mathscr{S}^{p\times q}(\Omega_+) \qquad \text{and} \qquad I_p - s(\mu)\, s(\mu)^* > 0 \qquad \text{a.e. on } \Omega_0. \tag{6.20}$$

This answer and the description of the set of all solutions to this problem may be obtained from the preceding analysis by passing from $W(\lambda)$ to $W(-\bar{\lambda})^*$. This formulation of the problem of the Darlington representation was considered earlier by [Ar1] and [De2].

The problem of finding $W \in \mathscr{U}(j_{pq})$ such that $T_W[0] = s$, for a given $s \in \mathscr{S}^{p\times q}(\Omega_+)$ is also equivalent to finding an $m \times m$ inner mvf

$$S(\lambda) = \begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{bmatrix}$$

with

$$s_{12}(\lambda) = s(\lambda) \qquad \text{and} \qquad \det\{s_{22}(\lambda)\} \not\equiv 0 \qquad \text{in } \Omega_+. \tag{6.21}$$

Each such $S(\lambda)$ is the Potapov–Ginzburg transform of a mvf $W \in \mathscr{U}(j_{pq})$ with $T_W[0] = s$. This formulation was considered without the restriction $\det\{s_{22}(\lambda)\} \not\equiv 0$ in Ω_+ by Belevitch [Be] in the rational case, and by Arov [Ar1], Dewilde [De1] and Douglas–Helton [DoH] in the general setting.

Another byproduct of the preceding analysis is a characterization of those mvf's $\mathfrak{A} \in \mathfrak{M}(p, q)$ which can arise in representations of the form (6.1) for $W \in \mathscr{U}(j_{pq})$:

THEOREM 6.1. *Let $\mathfrak{A} \in \mathfrak{M}(p, p)$. Then the following statements are equivalent*:

1. *There exist a pair of inner mvf's $b_1 \in \mathscr{S}_{in}^{p\times p}(\Omega_+)$ and $b_2 \in \mathscr{S}_{in}^{q\times q}(\Omega_+)$ such that condition* (6.2) *is met.*
2. $\mathfrak{A} \in \mathscr{N}^{m\times m}(\Omega_+)$.
3. $\mathfrak{A} \in \Pi \cap \mathscr{N}^{m\times m}(\Omega_+)$.
4. $\chi \in \Pi \cap \mathscr{S}^{q\times p}(\Omega_+)$.
5. $T_{\mathfrak{A}}[\mathscr{E}] \in \mathscr{N}^{p\times q}(\Omega_+)$ *for at least one* $\mathscr{E} \in \mathscr{S}^{p\times q}(\Omega_+)$.
6. $T_{\mathfrak{A}}[\mathscr{E}] \in \mathscr{N}^{p\times q}(\Omega_+)$ *for every* $\mathscr{E} \in \mathscr{S}^{p\times q}(\Omega_+)$.

If any one of these conditions hold, then for any given $f \in T_{\mathfrak{A}}[\mathcal{S}^{p\times q}]$, *there exist a pair of inner mvf's* $b_1 \in \mathcal{S}_{in}^{p\times p}(\Omega_+)$ *and* $b_2 \in \mathcal{S}_{in}^{q\times q}(\Omega_+)$ *such that* $b_1 f b_2 \in \mathcal{S}^{p\times q}(\Omega_+)$. *Moreover, for any such pair* $\{b_1, b_2\}$ *the mvf*

$$W(\lambda) = \begin{bmatrix} b_1(\lambda) & 0 \\ 0 & b_2(\lambda)^{-1} \end{bmatrix} \mathfrak{A}(\lambda)$$

belongs to $\mathcal{U}(j_{pq})$ *and* $\{b_1, b_2\} \in ap(W)$.

6.2. *A second parametrization of* j_{pq}*-inner mvf's when* $q = p$. When $q = p$, a second parametrization may be obtained from formula (6.19) by setting

$$\varphi_-(\mu) = \mathfrak{a}_-(\mu)\{I_p - \chi(\mu)^*\} \quad \text{and} \quad \varphi_+(\mu) = \mathfrak{a}_+(\mu)\{I_p - \chi(\mu)\} \tag{6.22}$$

a.e. on Ω_0 and

$$c(\lambda) = \{I_p + \chi(\lambda)\}\{I_p - \chi(\lambda)\}^{-1} \tag{6.23}$$

for $\lambda \in \Omega_+$. Then

$$c \in \Pi \cap \mathcal{C}^{p\times p}(\Omega_+) \qquad \text{and} \qquad c(\mu) + c(\mu)^* > 0 \qquad \text{a.e. on } \Omega_0. \tag{6.24}$$

Moreover, $c(\lambda)$ can be expressed directly in terms of the block entries of $W(\lambda)$ by the formula

$$c(\lambda) = \{w_{21}(\lambda) + w_{22}(\lambda)\}^{-1}\{w_{22}(\lambda) - w_{21}(\lambda)\} \tag{6.25}$$

and

$$\Delta(\mu) = \varphi_-(\mu)^*\,\varphi_-(\mu) = \varphi_+(\mu)^*\,\varphi_+(\mu) = 2\,\{c(\mu) + c(\mu)^*\}^{-1} \tag{6.26}$$

a.e. on Ω_0, is the boundary value of the mvf

$$\begin{aligned} \Delta(\lambda) &= \{w_{11}^{\#}(\lambda) + w_{12}^{\#}(\lambda)\}\{w_{11}(\lambda) + w_{12}(\lambda)\} \\ &= \{w_{21}^{\#}(\lambda) + w_{22}^{\#}(\lambda)\}\{w_{21}(\lambda) + w_{22}(\lambda)\}, \end{aligned} \tag{6.27}$$

where the equality between the two given expressions follows easily from the formula

$$[I_p \quad -I_p]\; j_p W^{\#}(\lambda)\; j_p W(\lambda) \begin{bmatrix} I_p \\ I_p \end{bmatrix} = 0.$$

Then the mvf's $\varphi_-(\lambda)$, $\varphi_+(\lambda)$ and $\Delta(\lambda)$ all belong to the class $\Pi \cap \mathcal{N}^{p\times p}(\Omega_+)$. Thus, we may extend the identity (6.26) from Ω_0 into $\mathbb{C}\backslash\Omega_0$ as follows:

$$\Delta(\lambda) := 2\{c(\lambda) + c^{\#}(\lambda)\}^{-1} = \varphi_-^{\#}(\lambda)\,\varphi_-(\lambda) = \varphi_+^{\#}(\lambda)\,\varphi_+(\lambda). \tag{6.28}$$

We can now formulate the following theorem:

THEOREM 6.2. *Let $W \in \mathcal{U}(j_p)$, let $\{b_1, b_2\} \in ap(W)$, let $c(\lambda)$ be defined in terms of the blocks $w_{ij}(\lambda)$ of $W(\lambda)$ by formula* (6.25). *Then $c(\lambda)$ meets the conditions in* (6.24) *and the mvf's*

$$\varphi_-(\lambda) = b_1(\lambda)^{-1}\,\{w_{11}(\lambda) + w_{12}(\lambda)\} \qquad \textit{and}$$
$$\varphi_+(\lambda) = b_2(\lambda)\{w_{21}(\lambda) + w_{22}(\lambda)\} \tag{6.29}$$

are such that:

1. *The identities in* (6.28) *hold.*
2. $\varphi_+ \in \Pi \cap \mathcal{N}_{out}^{p\times p}(\Omega_+)$ *and* $\varphi_-^{\#} \in \Pi \cap \mathcal{N}_{out}^{p\times p}(\Omega_+)$.
3. $(\rho_\omega \varphi_-^{\#})^{-1} \in H_2^{p\times p}(\Omega_+)$ *and* $(\rho_\omega \varphi_+)^{-1} \in H_2^{p\times p}(\Omega_+)$ *for at least one* (*and hence every*) *point* $\omega \in \Omega_+$.
4. $2(\varphi_-^{\#})^{-1}\,(I_p + c)^{-1} \in \mathcal{S}_{out}^{p\times p}(\Omega_+)$ *and* $2(I_p + c)^{-1}\,\varphi_+^{-1} \in \mathcal{S}_{out}^{p\times p}(\Omega_+)$.

Moreover,

$$W(\lambda) = \tfrac{1}{2}\begin{bmatrix} b_1(\lambda) & 0 \\ 0 & b_2(\lambda)^{-1} \end{bmatrix}\begin{bmatrix} \varphi_-(\lambda)\{I_p + c^{\#}(\lambda)\} & \varphi_-(\lambda)\{I_p - c^{\#}(\lambda)\} \\ \varphi_+(\lambda)\{I_p - c(\lambda)\} & \varphi_+(\lambda)\{I_p + c(\lambda)\} \end{bmatrix}. \tag{6.30}$$

Proof. The stated conclusions have all been established earlier in this section in terms of $\chi(\lambda)$ and are easily transposed to the needed form with the help of the identities (5.11), (5.21), (5.19) and

$$\mathfrak{a}_-(\mu) = \varphi_-(\mu)\left\{\frac{I_p + c^{\#}(\mu)}{2}\right\} \quad \text{and} \quad \mathfrak{a}_+(\mu) = \varphi_+(\mu)\left\{\frac{I_p + c(\mu)}{2}\right\}. \quad \blacksquare \tag{6.31}$$

Parametrization formulas closely related to the form (6.30) for $W \in \mathcal{U}(j_p)$ were established in [DeD] for the case $b_2(\lambda) = \text{constant}$.

Our next main objective is to show that the mvf which is obtained by replacing $c(\lambda)$ by $c_a(\lambda)$ in formula (6.30) is still j_p-inner. This will be an easy consequence of the next theorem which focuses on finer connections between $\mathfrak{A} \in \mathfrak{M}(p, p)$ and $\Delta(\mu)$ and is of independent interest.

THEOREM 6.3. *Let $\Delta(\mu)$ be a $p \times p$ mvf which meets the following three conditions*:

$$\Delta \in \Pi \cap \mathcal{N}^{p\times p}(\Omega_+),\ \Delta(\mu) > 0 \text{ a.e. on } \Omega_0 \quad \text{and} \quad \Delta^{-1} \in \widetilde{L_1^{p\times p}}(\Omega_0). \tag{6.32}$$

Let $\varphi_-(\lambda)$ and $\varphi_+(\lambda)$ be the essentially unique solutions of the factorization problem (6.26) *such that*

$$\varphi_-^{\#} \in \mathcal{N}_{out}^{p\times p}(\Omega_+) \qquad \text{and} \qquad \varphi_+ \in \mathcal{N}_{out}^{p\times p}(\Omega_+).$$

Then there exist uncountable many pairs of inner mvf's $b_1 \in \mathcal{S}_{in}^{p\times p}(\Omega_+)$ and $b_2 \in \mathcal{S}_{in}^{p\times p}(\Omega_+)$ such that

$$b_1 \varphi_- \varphi_+^{-1} b_2 \in \mathcal{S}_{in}^{p\times p}(\Omega_+). \tag{6.33}$$

Moreover, if $\mathfrak{A}_a(\mu)$ is defined in terms of $\varphi_-(\mu)$, $\varphi_+(\mu)$ and $c_a(\mu)$ by formulas (5.29) *and* (5.23), *and if $\{b_1, b_2\}$ is any pair of $p \times p$ inner mvf's for which* (6.33) *holds, then*:

1. $\mathfrak{A}_a \in \Pi \cap \mathcal{N}^{m\times m}(\Omega_+)$.
2. *The mvf*

$$W_a(\lambda) = \begin{bmatrix} b_1(\lambda) & 0 \\ 0 & b_2(\lambda)^{-1} \end{bmatrix} \mathfrak{A}_a(\lambda) \tag{6.34}$$

is j_p-inner.

3. $\{b_1, b_2\} \in ap(W_a)$.

Proof. The proof is divided into four steps.

Step 1. $\mathfrak{A}_a \in \Pi \cap \mathcal{N}^{m\times m}(\Omega_+)$.

Proof of Step 1. In view of formula (5.29), it suffices to show that the mvf's $c_a(\mu)$, $\varphi_-(\mu)$ and $\varphi_+(\mu)$ all belong to $\Pi \cap \mathcal{N}^{p\times p}(\Omega_+)$. But this is immediate from the assumptions (6.32) on $\Delta(\lambda)$, and the identities

$$\varphi_-(\mu) = \varphi_-^{\#}(\mu)^{-1}\, \Delta(\mu),$$

$$\varphi_+^{\#}(\mu) = \Delta(\mu)\, \varphi_+(\mu)^{-1} \qquad \text{and} \qquad c_a^{\#}(\mu) = 2\Delta(\mu)^{-1} - c_a(\mu),$$

which are valid a.e. on Ω_0.

Step 2.

$$2\{(I_p + c_a)\,\varphi_-^{\#}\}^{-1} \in \Pi \cap \mathcal{S}_{out}^{p\times p}(\Omega_+) \qquad \text{and}$$

$$2\{\varphi_+(I_p + c_a)\}^{-1} \in \Pi \cap \mathcal{S}_{out}^{p\times p}(\Omega_+). \tag{6.35}$$

Proof of Step 2. Let

$$\chi_a(\lambda) = \{I_p - c_a(\lambda)\}\{I_p + c_a(\lambda)\}^{-1}.$$

Then the identities (5.11), (5.21) and (5.19) hold with c_a and χ_a in place of c and χ respectively. Therefore, since

$$\{\varphi_-(\mu)^* \varphi_-(\mu)\}^{-1} = \frac{c(\mu)+c(\mu)^*}{2} = \frac{c_a(\mu)+c_a(\mu)^*}{2}$$

a.e. on Ω_0, the identity

$$\begin{aligned}2\{I_p + c_a(\mu)^*\}^{-1} \varphi_-(\mu)^{-1} (\varphi_-(\mu)^*)^{-1} 2\{I_p + c_a(\mu)\}^{-1} \\ = I_p - \chi_a(\mu)^* \chi_a(\mu)\end{aligned}$$

follows from (5.11) and (5.21). Thus the mvf

$$2\{(I_p + c_a)\varphi_-^{\#}\}^{-1} \in \Pi \cap \mathcal{N}_+^{p\times p}(\Omega_+)$$

is contractive a.e. on Ω_0 and hence belongs to $\Pi \cap \mathcal{S}_{out}^{p\times p}(\Omega_+)$.

The proof of the second assertion is established in much the same way via formulas (5.11) and (5.21) (with c_a in place of c and χ_a in place of χ) starting from

$$\{\varphi_+(\mu)^* \varphi_+(\mu)\}^{-1} = \frac{c(\mu)+c(\mu)^*}{2} = \frac{c_a(\mu)+c_a(\mu)^*}{2}$$

a.e. on Ω_0.

Step 3. $W_a \in \mathcal{U}(j_p)$.

Proof of Step 3. $W_a \in \mathcal{U}(j_p)$ if and only if the Potapov–Ginzburg transform $S_a(\lambda)$ of W_a belongs to $\mathcal{S}_{in}^{m\times m}(\Omega_+)$. Since $W_a(\mu)$ is j_p-unitary a.e. on Ω_0, $S_a(\mu)$ is automatically unitary a.e. on Ω_0 and therefore it suffices to show that $S_a \in \mathcal{N}_+^{m\times m}(\Omega_+)$. Let $[s_{ij}(\lambda)]$, $i, j = 1, 2$, denote the block decomposition of $S_a(\lambda)$. Then,

$$\begin{aligned}
s_{11}(\lambda) &= 2b_1(\lambda)\{\varphi_-^{\#}(\lambda)\}^{-1}\{I_p + c_a(\lambda)\}^{-1} \\
s_{12}(\lambda) &= b_1(\lambda)\varphi_-(\lambda)\{I_p - c_a^{\#}(\lambda)\}\{I_p + c_a(\lambda)\}^{-1}\varphi_+(\lambda)^{-1} b_2(\lambda) \\
&= b_1(\lambda)\varphi_-(\lambda)\varphi_+(\lambda)^{-1} b_2(\lambda) \\
&\quad - 2b_1(\lambda)\{\varphi_-^{\#}(\lambda)\}^{-1}\{I_p + c_a(\lambda)\}^{-1}\varphi_+(\lambda)^{-1} b_2(\lambda) \\
s_{21}(\lambda) &= \chi_a(\lambda) \\
s_{22}(\lambda) &= 2\{I_p + c_a(\lambda)\}^{-1}\varphi_+(\lambda)^{-1} b_2(\lambda).
\end{aligned}$$

In view of assumption (6.33) and Step 2, it is readily checked that $s_{ij} \in \mathcal{N}_+^{p\times p}(\Omega_+)$ for $i, j=1, 2$. Therefore, the full Potapov–Ginzburg transform $S_a(\lambda)$ of $W_a(\lambda)$ belongs to $\mathcal{N}_+^{m\times m}(\Omega_+)$, as needed.

Step 4. $\{b_1, b_2\} \in ap(W_a)$.

Proof of Step 4. Let w_{jj}, $j=1, 2$, denote the diagonal blocks of $W_a(\lambda)$. Then, in terms of the usual notation (5.29) for the block decomposition of $\mathfrak{A}_a(\lambda)$, we see that

$$w_{11}^{\#}(\lambda)^{-1} = b_1(\lambda)\, \mathfrak{a}_-^{\#}(\lambda)^{-1} \qquad \text{and} \qquad w_{22}(\lambda)^{-1} = \mathfrak{a}_+(\lambda)^{-1}\, b_2(\lambda).$$

Therefore, since $(\mathfrak{a}_-^{\#})^{-1}$ and $(\mathfrak{a}_+)^{-1}$ are both of class $\mathcal{S}_{out}^{p\times p}(\Omega_+)$, $\{b_1, b_2\} \in ap(W)$. ∎

THEOREM 6.4. *Let $W \in \mathcal{U}(j_p)$, let $\{b_1, b_2\} \in ap(W)$, let $c(\lambda)$, $\varphi_-(\lambda)$ and $\varphi_+(\lambda)$ be defined as in Theorem* 6.2 *and let $c_a(\lambda)$ be defined in Ω_+ by formula* (5.23). *Then*

$$c_s(\lambda) = c(\lambda) - c_a(\lambda)$$

belongs to $\Pi \cap \mathcal{C}^{p\times p}(\Omega_+)$;

$$c_s(\lambda) + c_s^{\#}(\lambda) = 0, \tag{6.36}$$

$$W(\lambda) = W_a(\lambda)\, W_s(\lambda), \tag{6.37}$$

where

$$W_a(\lambda) = \tfrac{1}{2}\begin{bmatrix} b_1(\lambda) & 0 \\ 0 & b_2(\lambda)^{-1} \end{bmatrix} \times \begin{bmatrix} \varphi_-(\lambda)\{I_p + c_a^{\#}(\lambda)\} & \varphi_-(\lambda)\{I_p - c_a^{\#}(\lambda)\} \\ \varphi_+(\lambda)\{I_p - c_a(\lambda)\} & \varphi_+(\lambda)\{I_p + c_a(\lambda)\} \end{bmatrix} \tag{6.38}$$

and

$$W_s(\lambda) = I_{2p} + \tfrac{1}{2}\begin{bmatrix} -c_s(\lambda) & c_s(\lambda) \\ -c_s(\lambda) & c_s(\lambda) \end{bmatrix}. \tag{6.39}$$

Moreover, $W_a \in \mathcal{U}(j_p)$, $W_s \in \mathcal{U}_S(j_p)$ and the following statements are equivalent:

1. $W \in \mathcal{U}_{rR}(j_p)$.
2. $c_s(\lambda)$ *is constant and* $W_a \in \mathcal{U}_{rR}(j_p)$.
3. $c_s(\lambda)$ *is constant and index* $\{b_1^{-1} T_W[I_p]\, b_2^{-1}\} = 0$.

Proof. Let $\Delta(\lambda)$ be defined in terms of the blocks of the given mvf $W \in \mathcal{U}(j_{pq})$ by formula (6.27). Then it is readily checked that $\Delta(\lambda)$ meets the three conditions in (6.32) and hence, by Theorem 6.3, that $W_a \in \mathcal{U}(j_p)$. The fact that $W_s \in \mathcal{U}_S(j_p)$ was established in Theorem 5.2. It remains only to verify that the final three statements (1)–(3) are equivalent. By definition

$$W \in \mathcal{U}_{rR}(j_p) \Leftrightarrow W_s(\lambda) \text{ is constant and } W_a \in \mathcal{U}_{rR}(j_p)$$
$$\Leftrightarrow W_s(\lambda) \text{ is constant and } \mathfrak{A}_a \in \mathfrak{M}_R(p, p).$$

By Theorem 5.4,

$$\mathfrak{A}_a \in \mathfrak{M}_R(p, p) \Leftrightarrow \operatorname{index}\{T_{\mathfrak{A}}[I_p]\} = 0.$$

The rest is plain, since

$$\operatorname{index}\{T_{\mathfrak{A}}[I_p]\} = \operatorname{index}\{b_1^{-1} T_W[I_p]\, b_2^{-1}\}$$

and, by formula (6.39),

$$W_s(\lambda) \text{ is constant} \Leftrightarrow c_s(\lambda) \text{ is constant.} \quad \blacksquare$$

The equivalence of (1) and (3) was established earlier in [Ar9] under some extra normalization conditions, that are now seen to be superfluous.

Formula (6.30) displays the fact that $W \in \mathcal{U}(j_p)$ is parametrized by a $p \times p$ mvf $c(\lambda)$ that meets the conditions (6.24) and $\{b_1, b_2\} \in ap(W)$. The $p \times p$ mvf's $\varphi_-(\lambda)$ and $\varphi_+(\lambda)$ are defined essentially uniquely by $c(\lambda)$ through the factorization formulas in (6.26) and the requirement that φ_+ and $\varphi_-^{\#}$ belong to the class $\mathcal{N}_{out}^{p \times p}(\Omega_+)$. The identity

$$b_1 \varphi_- \varphi_+^{-1} b_2 = T_W[I_p] \tag{6.40}$$

guarantees that

$$b_1 \varphi_- \varphi_+^{-1} b_2 \in \mathcal{S}_{in}^{p \times p}(\Omega_+). \tag{6.41}$$

This condition serves to characterize the class of $\mathfrak{A} \in \mathfrak{M}_R(p, p)$ which can arise in the parametrization of $W \in \mathcal{U}(j_p)$ through the formula

$$W(\mu) = \begin{bmatrix} b_1(\mu) & 0 \\ 0 & b_2(\mu)^{-1} \end{bmatrix} \mathfrak{A}(\mu)$$

a.e. on Ω_0. The condition (6.41) is a special case of the general fact that

$$b_1 T_{\mathfrak{A}}[\mathcal{S}^{p \times q}]\, b_2 \subset \mathcal{S}^{p \times q}(\Omega_+). \tag{6.42}$$

In fact (6.41) is equivalent to (6.42), as we have already noted in Theorem 6.1.

THEOREM 6.5. *Let $c(\lambda)$ be a $p \times p$ mvf that meets the conditions in* (6.24) *and let $\varphi_-(\lambda)$ and $\varphi_+(\lambda)$ be the essentially unique solutions of the factorization problem* (6.26) *such that $\varphi_-^{\#}(\lambda)$ and $\varphi_+(\lambda)$ both belong to $\mathcal{N}_{out}^{p\times p}(\Omega_+)$. Then there exist uncountably many pairs of inner mvf's $b_1, b_2 \in \mathcal{S}_{in}^{p\times p}(\Omega_+)$ such that*

$$b_1 \varphi_- \varphi_+^{-1} b_2 \in \mathcal{S}_{in}^{p\times p}(\Omega_+). \tag{6.43}$$

Moreover, for every such pair $\{b_1, b_2\}$, the mvf $W(\lambda)$ defined by formula (6.30) *belongs to $\mathcal{U}(j_p)$ and $\{b_1, b_2\} \in ap(W)$.*

Proof. If $c(\lambda)$ meets the two conditions in (6.24), then

$$\Delta(\lambda) = 2\{c(\lambda) + c^{\#}(\lambda)\}^{-1}$$

meets the three conditions in (6.32). Therefore, Theorem 6.3 is applicable and guarantees that the mvf $W_a(\lambda)$ defined by formula (6.38) belongs to $\mathcal{U}(j_p)$ and that $\{b_1, b_2\} \in ap(W_a)$. The factor $W_s(\lambda) = W_a(\lambda)^{-1} W(\lambda)$ belongs to $\mathcal{U}_S(j_p)$ by the argument in Theorem 5.2. Therefore, $W \in \mathcal{U}(j_p)$ and, since the multiplication of $W_a(\lambda)$ on the right by a singular j_p-inner mvf does not change the associated pairs, i.e.,

$$\{b_1, b_2\} \in ap(W_a) \Leftrightarrow \{b_1, b_2\} \in ap(W_a W_s),$$

the proof is complete. ▮

THEOREM 6.6. *Let $W \in \mathcal{U}(j_p)$ and let $c(\lambda)$, $\Delta(\lambda)$, $\varphi_-(\lambda)$ and $\varphi_+(\lambda)$ be defined by formulas* (6.25) *and* (6.28) *and the requirement that $\varphi_-^{\#}(\lambda)$ and $\varphi_+(\lambda)$ belong to $\mathcal{N}_{out}^{p\times p}(\Omega_+)$. Then the following statements are equivalent*:

1. $W \in \mathcal{U}_{sR}(j_p)$.

2. *$\Delta(\mu)$ satisfies the matricial Muckenhoupt condition* (4.2) *and $W \in \widetilde{L_2^{m\times m}}(\Omega_0)$.*

3. *$\Delta(\mu)$ satisfies the matricial Muckenhoupt condition* (4.2) *and $c_s(\lambda)$ is constant.*

4. *$\Delta \in \widetilde{L_1^{p\times p}}(\Omega_0)$, the Hankel operator Γ_g based on $g = \varphi_- \varphi_+^{-1}$ is strictly contractive and $c_s(\lambda)$ is constant.*[3]

[3] For other conditions in terms of Toeplitz-like operators, see Theorem 6.6 of [ArD1].

Proof. Let $\{b_1, b_2\} \in ap(W)$. Then

$$\mathfrak{A}(\lambda) = \begin{bmatrix} b_1(\lambda) & 0 \\ 0 & b_2(\lambda)^{-1} \end{bmatrix}^{-1} W(\lambda) \tag{6.44}$$

belongs to $\mathfrak{M}(p, p)$ and

$$W \in \mathcal{U}_{sR}(j_p) \Leftrightarrow \mathfrak{A} \in \mathfrak{M}_{sR}(p, p).$$

The rest is immediate from Theorems 4.8 and 5.5. ∎

Let $\Delta(\mu)$ be a measurable $p \times p$ mvf on Ω_0 that is positive semidefinite a.e., let

$$\begin{aligned} \mathcal{W}_\Delta = \{ W \in \mathcal{U}(j_p) : \{w_{11}(\mu) + w_{12}(\mu)\}^* \{w_{11}(\mu) + w_{12}(\mu)\} \\ = \Delta(\mu) \text{ a.e. on } \Omega_0\}. \end{aligned} \tag{6.45}$$

To this point we have shown that:

1. $\mathcal{W}_\Delta \neq \emptyset \Leftrightarrow \Delta$ satisfies the three conditions in (6.32).
2. $\mathcal{W}_\Delta \cap \mathcal{U}_{rR}(j_p) \neq \emptyset \Leftrightarrow$

$$\begin{aligned} &\Delta \text{ satisfies the three conditions in (6.32) and index } \{T_{\mathfrak{A}_a}(I_p)\} = 0 \\ &\text{for the mvf } \mathfrak{A}_a(\mu) \text{ defined by } \Delta(\mu) \text{ via formula (5.29).} \end{aligned} \tag{6.46}$$

3. $\mathcal{W}_\Delta \cap \mathcal{U}_{sR}(j_p) \neq \emptyset \Leftrightarrow$

$$\begin{aligned} &\Delta \text{ satisfies the three conditions in (6.32) and} \\ &\text{meets the matricial Muckenhoupt condition (4.2).} \end{aligned} \tag{6.47}$$

In fact, it is now easy to write down a complete description of the sets $\mathcal{W}_\Delta$, $\mathcal{W}_\Delta \cap \mathcal{U}_{rR}(j_p)$ and $\mathcal{W}_\Delta \cap \mathcal{U}_{sR}(j_p)$.

THEOREM 6.7. *Let $\Delta(\lambda)$ be a $p \times p$ mvf which meets the three conditions in* (6.32). *Then $W \in \mathcal{W}_\Delta$ if and only if it can be expressed in the form*

$$W(\lambda) = \begin{bmatrix} b_1(\lambda) & 0 \\ 0 & b_2(\lambda)^{-1} \end{bmatrix} \mathfrak{A}_a(\lambda)\, \mathfrak{A}_s(\lambda), \tag{6.48}$$

where the factor $\mathfrak{A}_a \in \mathfrak{M}(p, p)$ and is uniquely determined by $\Delta(\lambda)$ up to a left constant block diagonal unitary factor via formula (5.29), *$\mathfrak{A}_s \in \mathfrak{M}_s(p, p)$ is defined by formula* (5.30) *for any $c_s \in \mathcal{C}^{p\times p}$ with $(\Re c_s)(\mu) = 0$ a.e. on Ω_0 and $\{b_1, b_2\}$ is any pair of mvf's in $\mathcal{S}_{in}^{p\times p}(\Omega_+)$ such that*

$$b_1 T_{\mathfrak{A}_a}[I_p]\, b_2 \in \mathcal{S}_{in}^{p\times p}(\Omega_+). \tag{6.49}$$

Moreover, we have the following supplementary conclusions:

1. *If* Δ *satisfies the four conditions in* (6.46), *then* $W \in \mathscr{W}_\Delta \cap \mathscr{U}_{rR}(j_p)$ *if and only if it can be expressed in the general form* (6.48) *and* $\mathfrak{A}_s(\lambda)$ *is constant, i.e.*,

$$\mathfrak{A}_s(\lambda) = \begin{bmatrix} I_p - i\gamma & i\gamma \\ -i\gamma & I_p + i\gamma \end{bmatrix}. \tag{6.50}$$

2. *If* Δ *satisfies the four conditions in* (6.47), *then* $W \in \mathscr{W}_\Delta \cap \mathscr{U}_{sR}(j_p)$ *if and only if it can be expressed in the general form* (6.48) *and* $\mathfrak{A}_s(\lambda)$ *constant, i.e.*, $\mathfrak{A}_s(\lambda)$ *is given by* (6.50).

7. PARAMETRIZATION FOR THE CLASSES $\mathscr{U}(J_p)$ AND $\mathscr{U}(\mathscr{J}_p)$, DUALITY, AND AN EXAMPLE

In the first part of this section, a few of the main conclusions that were obtained earlier for the class $\mathscr{U}(j_p)$ will be reformulated for the classes $\mathscr{U}(J_p)$ and $\mathscr{U}(\mathscr{J}_p)$ that are based on the signature matrices

$$J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix} \quad \text{and} \quad \mathscr{J}_p = \begin{bmatrix} 0 & iI_p \\ -iI_p & 0 \end{bmatrix}. \tag{7.1}$$

Both of these classes are important for applications. Subsequently, in the second subsection, a dual set of parametrization formulas that are appropriate for the study of left regularity, will be discussed briefly. This is followed by a short detour on left γ-generating mvf's. We turn next to the case of entire J-inner mvf's. Some general tests for checking when $c_s(\lambda)$ (and its analogue $z_s(\lambda)$) are constant, are then taken up in a fifth subsection. Finally, a one-parameter family of strongly regular entire J-inner mvf's that are unbounded on $\mathbb{R}$ is presented.

7.1. *Parametrizations and conclusions for* J_p *and* $\mathscr{J}_p$*-inner mvf's.* It is easy to reformulate the conclusions that were obtained for the signature matrix j_p to the other signature matrices of interest, because if M is a unitary matrix such that $M^*JM = j_p$, then:

$$U \in \mathscr{U}(J) \Leftrightarrow M^*UM \in \mathscr{U}(j_p),$$

$$U \in \mathscr{U}_S(J) \Leftrightarrow M^*UM \in \mathscr{U}_S(j_p),$$

$$U \in \mathscr{U}_{rR}(J) \Leftrightarrow M^*UM \in \mathscr{U}_{rR}(j_p),$$

$$U \in \mathscr{U}_{sR}(J) \Leftrightarrow M^*UM \in \mathscr{U}_{sR}(j_p).$$

TABLE 7.1

J	j_p	J_p	$\mathscr{J}_p$
$c(\lambda)$	$(u_{22}+u_{21})^{-1}(u_{22}-u_{21})$	$(u_{12}+u_{22})^{-1}(u_{11}+u_{21})$	$(u_{22}+iu_{12})^{-1}(u_{11}-iu_{21})$
---	------------	------------	------------
$=$	$(u_{11}^{\#}-u_{12}^{\#})(u_{11}^{\#}+u_{12}^{\#})^{-1}$	$(u_{11}^{\#}-u_{21}^{\#})(u_{22}^{\#}-u_{12}^{\#})^{-1}$	$(u_{11}^{\#}-iu_{21}^{\#})(u_{22}^{\#}+iu_{12}^{\#})^{-1}$
$G(\lambda)$	$u_{11}+u_{12}$	$u_{22}-u_{12}$	$u_{22}-iu_{12}$
$H(\lambda)$	$u_{21}+u_{22}$	$u_{22}+u_{12}$	$u_{22}+iu_{12}$
$G^{\bullet}(\lambda)=$ $G(\lambda)\,c^{\#}(\lambda)$	$u_{11}-u_{12}$	$u_{11}-u_{21}$	$u_{11}+iu_{21}$
$H^{\bullet}(\lambda)=$ $H(\lambda)\,c(\lambda)$	$u_{22}-u_{21}$	$u_{11}+u_{21}$	$u_{11}-iu_{21}$

Thus, as

$$\mathfrak{C}=\frac{1}{\sqrt{2}}\begin{bmatrix}-I_p & I_p\\ I_p & I_p\end{bmatrix} \quad\text{and}\quad \mathfrak{D}=\frac{1}{\sqrt{2}}\begin{bmatrix}iI_p & -iI_p\\ I_p & I_p\end{bmatrix} \tag{7.2}$$

are unitary matrices such that

$$\mathfrak{C}J_p\mathfrak{C}=j_p \qquad \mathfrak{D}^*\mathscr{J}_p\mathfrak{D}=j_p, \tag{7.3}$$

it remains only to calculate and tabulate the results in a convenient form. Table 7.1 serves to define the $p\times p$ mvf's $c(\lambda)$, $G(\lambda)$, $H(\lambda)$,

$$G^{\bullet}(\lambda)=G(\lambda)\,c^{\#}(\lambda) \quad\text{and}\quad H^{\bullet}(\lambda)=H(\lambda)\,c(\lambda) \tag{7.4}$$

in terms of the block entries $u_{ij}(\lambda)$, $i, j=1, 2$, of a mvf $U\in\mathscr{U}(J)$ for $J=j_p$, J_p and $\mathscr{J}_p$.

THEOREM 7.1. *Let $U\in\mathscr{U}(J)$ and let $c(\lambda)$, $G(\lambda)$, $H(\lambda)$, $G^{\bullet}(\lambda)$ and $H^{\bullet}(\lambda)$ be the $p\times p$ mvf's that are defined in terms of the block entries $u_{ij}(\lambda)$ of $U(\lambda)$ in Table* 7.1 *for the indicated choices of the signature matrix J. Then*:

1.

$$c\in\Pi\cap\mathscr{C}^{p\times p}(\Omega_+) \qquad \textit{and} \qquad c(\mu)+c(\mu)^*>0 \qquad \textit{a.e. on } \Omega_0. \tag{7.5}$$

2. *The mvf*

$$\Delta(\lambda)=2\{c(\lambda)+c^{\#}(\lambda)\}^{-1} \tag{7.6}$$

meets the three conditions in (6.32).

3. *$G(\lambda)$ and $H(\lambda)$ are solutions of the factorization problem*

$$G(\mu)^*\,G(\mu)=H(\mu)^*\,H(\mu)=\Delta(\mu) \qquad \textit{a.e. on } \Omega_0. \tag{7.7}$$

TABLE 7.2

J	j_p	J_p	$\mathcal{J}_p$
$2U(\lambda)$	$\begin{bmatrix} G+G^{\bullet} & G-G^{\bullet} \\ H-H^{\bullet} & H+H^{\bullet} \end{bmatrix}$	$\begin{bmatrix} H^{\bullet}+G^{\bullet} & H-G \\ H^{\bullet}-G^{\bullet} & H+G \end{bmatrix}$	$\begin{bmatrix} H^{\bullet}+G^{\bullet} & i(G-H) \\ i(H^{\bullet}-G^{\bullet}) & H+G \end{bmatrix}$
$U_s(\lambda)$	$I_{2p}+\frac{1}{2}\begin{bmatrix} -c_s(\lambda) & c_s(\lambda) \\ -c_s(\lambda) & c_s(\lambda) \end{bmatrix}$	$\begin{bmatrix} I_p & 0 \\ c_s(\lambda) & I_p \end{bmatrix}$	$\begin{bmatrix} I_p & 0 \\ ic_s(\lambda) & I_p \end{bmatrix}$

4. *There exist an essentially unique pair of mvf's* $b_1 \in \mathcal{S}_{in}^{p\times p}(\Omega_+)$ *and* $b_2 \in \mathcal{S}_{in}^{p\times p}(\Omega_+)$ *such that*

$$G(\lambda) = b_1(\lambda)\,\varphi_-(\lambda) \qquad \text{and} \qquad H(\lambda) = b_2(\lambda)^{-1}\,\varphi_+(\lambda), \tag{7.8}$$

where φ_- *and* φ_+ *are the essentially unique solutions of the factorization problem* (6.28) *such that*

$$(\varphi_-^{\#})^{\pm 1} \in \Pi \cap \mathcal{N}_+^{p\times p}(\Omega_+) \qquad \text{and} \qquad (\varphi_+)^{\pm 1} \in \Pi \cap \mathcal{N}_+^{p\times p}(\Omega_+). \tag{7.9}$$

Now let

$$g_\Delta(\lambda) = \varphi_-(\lambda)\,\varphi_+(\lambda)^{-1}, \tag{7.10}$$

let $c_a(\lambda)$ *be defined in terms of* $\Delta(\lambda)$ *by formula* (5.23) *and let*

$$c_s(\lambda) = c(\lambda) - c_a(\lambda). \tag{7.11}$$

Then:

5. $c_s \in \mathcal{C}^{p\times p}(\Omega_+)$ *and* $c_s(\mu) + c_s(\mu)^* = 0$ *a.e. on* Ω_0.
6.[4] $c_s \in \Pi \cap \mathcal{C}^{p\times p}(\Omega_+)$ *and* $c_s(\lambda) + c_s^{\#}(\lambda) = 0$.
7. $U(\lambda)$ *can be expressed in the form shown in Table* 7.2.
8.

$$U(\lambda) = U_a(\lambda)\,U_s(\lambda), \tag{7.12}$$

where $U_s \in \mathcal{U}_S(J)$ *has the form given in Table* 7.2 *and* $U_a \in \mathcal{U}(J)$ *is given by the formula for* $U(\lambda)$ *in Table* 7.2 *with the same* $G(\lambda)$ *and* $H(\lambda)$, *but with* $G^{\bullet}(\lambda)$ *and* $H^{\bullet}(\lambda)$ *replaced by*

$$G_a^{\bullet}(\lambda) = G(\lambda)\,c_a^{\#}(\lambda) \qquad \text{and} \qquad H_a^{\bullet}(\lambda) = H(\lambda)\,c_a(\lambda).$$

[4] Properties (5) and (6) are equivalent.

Moreover, we have the following sets of equivalences:

$$
\begin{aligned}
U \in \mathcal{U}_{rR}(J) &\Leftrightarrow c_s(\lambda) \textit{ is constant and } U_a \in \mathcal{U}_{rR}(J).\\
&\Leftrightarrow c_s(\lambda) \textit{ is constant and } \operatorname{index}\{g_\Delta\} = 0.\\
U \in \mathcal{U}_{sR}(J) &\Leftrightarrow U \in \widetilde{L_2^{m\times m}}(\Omega_0) \textit{ and } \Delta(\mu) \textit{ satisfies the matricial}\\
&\qquad \textit{Muckenhoupt condition } (4.2).\\
&\Leftrightarrow c_s(\lambda) \textit{ is constant and } U_a \in \mathcal{U}_{sR}(J).\\
&\Leftrightarrow c_s(\lambda) \textit{ is constant and } \Delta(\mu) \textit{ satisfies the matricial}\\
&\qquad \textit{Muckenhoupt condition } (4.2).\\
&\Leftrightarrow c_s(\lambda) \textit{ is constant, } \Delta \in \widetilde{L_2^{p\times p}}(\Omega_0) \textit{ and the Hankel}\\
&\qquad \textit{operator with symbol } g_\Delta \textit{ is strictly contractive.}
\end{aligned}
$$

To formulate the next result, it is convenient to let $c_U(\lambda)$, $G_U(\lambda)$ and $H_U(\lambda)$ denote the $p \times p$ mvf's that are defined in terms of the blocks of $U(\lambda)$ in Table 7.1, for each of the three considered signature matrices. (The dependence on J is not indicated explicitly in order to keep the notation reasonable.)

THEOREM 7.2. *Let $\Delta(\mu)$ be a $p \times p$ measurable mvf on Ω_0 that is positive definite a.e. and let $\Delta^{-1} \in \widetilde{L_1^{p\times p}}(\Omega_0)$. Let*

$$\mathcal{U}_\Delta = \mathcal{U}_\Delta(J) = \{U \in \mathcal{U}(J) : G_U(\mu)^* G_U(\mu) = \Delta(\mu) \text{ a.e. on } \Omega_0\}, \tag{7.13}$$

let $c_a(\lambda)$ and $g_\Delta(\lambda)$ be defined in terms of Δ by formulas (5.23) *and* (7.10), *respectively, let*

$$c_s(\lambda) = c_U(\lambda) - c_a(\lambda). \tag{7.14}$$

Then:

1. $\mathcal{U}_\Delta \neq \emptyset \Leftrightarrow \Delta$ *satisfies the three conditions in* (6.32).
2. $\mathcal{U}_\Delta \cap \mathcal{U}_{rR}(J) \neq \emptyset \Leftrightarrow \Delta$ *satisfies the three conditions in* (6.32) *and* $\operatorname{index}\{g_\Delta\} = 0$.
3. $\mathcal{U}_\Delta \cap \mathcal{U}_{sR}(J) \neq \emptyset \Leftrightarrow \Delta$ *satisfies the three conditions in* (6.32) *and Δ meets the matricial Muckenhoupt condition* (4.2).

We remark that the mvf

$$\Delta^\bullet(\lambda) = 2\{c(\lambda)^{-1} + c^\#(\lambda)^{-1}\}^{-1} \tag{7.15}$$

meets the same three conditions in (6.32) as $\Delta(\lambda)$. Moreover, the mvf's $G^{\bullet}(\lambda)$ and $H^{\bullet}(\lambda)$ are solutions of the factorization problem

$$G^{\bullet}(\mu)^* G^{\bullet}(\mu) = H^{\bullet}(\mu)^* H^{\bullet}(\mu) = \Delta^{\bullet}(\mu) \tag{7.16}$$

a.e. on Ω_0. Thus one can develop the theory in terms of these parameters instead of $G(\lambda)$ and $H(\lambda)$.

THEOREM 7.3. *Let $\Delta(\mu)$ be a $p \times p$ measurable mvf on Ω_0 that satisfies the three conditions in* (6.32). *Then the set $\mathcal{U}_\Delta(J)$ is described by Table* 7.2 *where the entries in the table are computed according to the following algorithm*:

1. *Obtain $\varphi_-(\lambda)$ and $\varphi_+(\lambda)$ as solutions of the factorization problem*

$$\varphi_-(\mu)^* \varphi_-(\mu) = \varphi_+(\mu)^* \varphi_+(\mu) = \Delta(\mu)$$

a.e. on Ω_0 such that

$$(\varphi_-^{\#})^{\pm 1} \in \Pi \cap \mathcal{N}_+^{p \times p}(\Omega_+) \qquad \text{and} \qquad (\varphi_+)^{\pm 1} \in \Pi \cap \mathcal{N}_+^{p \times p}(\Omega_+).$$

2. *Define $g_\Delta(\lambda) = \varphi_-(\lambda)\, \varphi_+(\lambda)^{-1}$.*
3. *Choose $b_1, b_2 \in \mathcal{S}_{in}^{p \times p}(\Omega_+)$ such that*

$$b_1 g_\Delta b_2 \in \mathcal{S}_{in}^{p \times p}(\Omega_+).$$

4. *Define $c_a(\lambda)$ by formula* (5.23), *which depends only on $\Delta(\mu)$.*
5. *Choose any mvf $c_s \in \Pi \cap \mathcal{C}^{p \times p}(\Omega_+)$ such that $c_s(\lambda) + c_s^{\#}(\lambda) = 0$.*
6. *Set $c(\lambda) = c_a(\lambda) + c_s(\lambda)$.*
7. *Set $G(\lambda) = b_1(\lambda)\, \varphi_-(\lambda)$, $G^{\bullet}(\lambda) = G(\lambda)\, c^{\#}(\lambda)$, $H(\lambda) = b_2(\lambda)^{-1}\, \varphi_+(\lambda)$, $H^{\bullet}(\lambda) = H(\lambda)\, c(\lambda)$.*

THEOREM 7.4. *Let $c(\lambda)$ be a $p \times p$ mvf that meets the two conditions in* (6.24). *Let $\Delta(\lambda)$ be defined by formula* (6.26). *Then the set of all mvf's $U \in \mathcal{U}(J)$ for which $c_U(\lambda) = c(\lambda)$ is described by Table* 7.2, *where the entries are computed by invoking Steps* (1), (2), (3) *and* (7) *of the algorithm presented in the previous theorem.*

7.2. *Dual parametrizations and left regularity.* The preceding sets of parametrization formulas are all derived from the parametrization of $W \in \mathcal{U}(j_{pq})$ in terms of $\{b_1, b_2\}$ and $\chi = -w_{22}^{-1} w_{21}$ that was considered in Subsection 6.1. This parametrization depends upon the fact that $\chi(\lambda)$ meets the conditions given in (6.9) and hence that, if $p = q$, the mvf $c(\lambda)$ defined by formula (6.23) meets the conditions in (6.24).

TABLE 7.3

J	j_p	J_p	$\mathscr{J}_p$
$z(\lambda)$	$(u_{12}+u_{22})\times(u_{22}-u_{12})^{-1}$	$(u_{21}+u_{22})\times(u_{11}+u_{12})^{-1}$	$(u_{22}-iu_{21})\times(u_{11}+iu_{12})^{-1}$
---	-------	-------	-------
$=$	$(u_{11}^*-u_{21}^*)^{-1}\times(u_{11}^*+u_{21}^*)$	$(u_{11}^*-u_{12}^*)^{-1}\times(u_{22}^*-u_{21}^*)$	$(u_{11}^*+iu_{12}^*)^{-1}\times(u_{22}^*-iu_{21}^*)$
$F(\lambda)$	$u_{11}-u_{21}$	$u_{11}-u_{12}$	$u_{11}-iu_{12}$
$E(\lambda)$	$u_{22}-u_{12}$	$u_{11}+u_{12}$	$u_{11}+iu_{12}$
$F^\bullet(\lambda)= z^\#(\lambda)\,F(\lambda)$	$u_{11}+u_{21}$	$u_{22}-u_{21}$	$u_{22}+iu_{21}$
$E^\bullet(\lambda)= z(\lambda)\,E(\lambda)$	$u_{12}+u_{22}$	$u_{21}+u_{22}$	$u_{22}-iu_{21}$
$2U(\lambda)$	$\begin{bmatrix} F^\bullet+F & E^\bullet-E \\ F^\bullet-F & E^\bullet+E \end{bmatrix}$	$\begin{bmatrix} E+F & E-F \\ E^\bullet-F^\bullet & E^\bullet+F^\bullet \end{bmatrix}$	$\begin{bmatrix} E+F & i(F-E) \\ i(E^\bullet-F^\bullet) & E^\bullet+F^\bullet \end{bmatrix}$

The parametrizations stemming from $\chi(\lambda)$ are ideally suited for the study of right regularity and "right" strong regularity. There is, however, an equally valid dual set of parametrizations that is suitable for the study of left regularity and "left" strong regularity. The starting point is the upper right hand corner of the Potapov–Ginzburg transform of $W(\lambda)$:

$$s_{12}(\lambda)=w_{12}(\lambda)\,w_{22}(\lambda)^{-1}=w_{11}^\#(\lambda)^{-1}\,w_{21}^\#(\lambda) \tag{7.17}$$

and, for $p=q$, the mvf

$$\begin{aligned} z(\lambda) &= \{I_p+s_{12}(\lambda)\}\{I_p-s_{12}(\lambda)\}^{-1} \\ &= \{w_{12}(\lambda)+w_{22}(\lambda)\}\{w_{22}(\lambda)-w_{12}(\lambda)\}^{-1} \\ &= \{w_{11}^\#(\lambda)-w_{21}^\#(\lambda)\}^{-1}\{w_{11}^\#(\lambda)+w_{21}^\#(\lambda)\}. \end{aligned} \tag{7.18}$$

The corresponding parametrizations of $U\in\mathscr{U}(J)$ are recorded in Table 7.3 for J equal to j_p, J_p and $\mathscr{J}_p$.

In this parametrization, the mvf

$$F(\lambda)\,F^\#(\lambda)=E(\lambda)\,E^\#(\lambda)=2\{z(\lambda)+z^\#(\lambda)\}^{-1}, \tag{7.19}$$

plays the role of $\varDelta(\lambda)$, whereas

$$F^\bullet(\lambda)(F^\bullet)^\#(\lambda)=E^\bullet(\lambda)(E^\bullet)^\#(\lambda)=2\{z(\lambda)^{-1}+z^\#(\lambda)^{-1}\}^{-1} \tag{7.20}$$

plays the role of $\varDelta^\bullet(\lambda)$.

Parametrizations of this second form were considered in [DeD], [DyI], [AlD1], [AlD2], [Dy2], [Dy3] in a variety of settings for assorted applications. The characteristic properties of the pairs $\{E, F\}$ and $\{G, H\}$ that appear in the parametrizations of mvf's in the class $\mathcal{U}(j_p)$ are discussed in the paper [FKM].

For $U \in \mathcal{U}(J)$ we define $U_{\#}(\lambda)$ by the rule

$$U_{\#}(\lambda) = \begin{cases} \mathcal{J}_p U^{\#}(\lambda)\, \mathcal{J}_p & \text{if } \quad J = j_p \text{ or } J = J_p \\ J_p U^{\#}(\lambda)\, J_p & \text{if } \quad J = \mathcal{J}_p. \end{cases} \tag{7.21}$$

It is readily checked that

$$U \in \mathcal{U}(J) \Leftrightarrow U_{\#} \in \mathcal{U}(J) \tag{7.22}$$

and

$$(U_{\#})_{\#}\,(\lambda) = U(\lambda) \tag{7.23}$$

for each of the considered choices of J.

A mvf $U \in \mathcal{U}(J)$ is said to belong to the class $\mathcal{U}_{\ell R}(J)$ of left regular J-inner mvf's if $U(\lambda)$ has no nonconstant singular left J-inner divisors, i.e., if $U(\lambda) = U_1(\lambda)\, U_2(\lambda)$, where $U_1 \in \mathcal{U}_S(J)$ and $U_2 \in \mathcal{U}(J)$, then $U_1(\lambda)$ is constant. It is easily checked that

$$U \in \mathcal{U}_{rR}(J) \Leftrightarrow U_{\#} \in \mathcal{U}_{\ell R}(J). \tag{7.24}$$

By analogy, we shall refer to the class $\mathcal{U}_{sR}(J)$ as the class of right strongly regular J-inner mvf's and shall say that $U \in \mathcal{U}(J)$ is left strongly regular if $U_{\#}$ is right strongly regular.

If $W \in \mathcal{U}(j_p)$ and $\mathcal{E} \in \mathcal{S}^{p \times p}(\Omega_+)$, then the linear fractional transformation

$$T_W[\mathcal{E}] = (w_{11}^{\#} + \mathcal{E} w_{12}^{\#})^{-1}\,(w_{21}^{\#} + \mathcal{E} w_{22}^{\#}) \tag{7.25}$$

(see e.g., Theorem 3.5 of [Dy1] applied to $j_p W^{\#} j_p = W^{-1}$). Thus

$$T_{W_{\#}}[\mathcal{E}] = (w_{11} - \mathcal{E} w_{12})^{-1}\,(-w_{21} + \mathcal{E} w_{11}) \tag{7.26}$$

and we see that $W \in \mathcal{U}(j_p)$ is left strongly regular if there exists an $\mathcal{E} \in \mathcal{S}^{p \times p}(\Omega_+)$ such that

$$\|(w_{22} + \mathcal{E} w_{12})^{-1}\,(w_{21} + \mathcal{E} w_{11})\|_{\infty} < 1. \tag{7.27}$$

It is now convenient to let $c_U(\lambda)$, $H_U(\lambda)$, $G_U(\lambda)$, $H_U^\bullet(\lambda)$ and $G_U^\bullet(\lambda)$ denote the formulas for $c(\lambda)$, $H(\lambda)$, $G(\lambda)$, $H^\bullet(\lambda)$ and $G^\bullet(\lambda)$ in terms of the blocks of $U(\lambda)$ that are given in Table 7.1 and, similarly, to let $z_U(\lambda)$, $E_U(\lambda)$, $F_U(\lambda)$, $E_U^\bullet(\lambda)$ and $F_U^\bullet(\lambda)$ to denote the formulas for $z(\lambda)$, $E(\lambda)$, $F(\lambda)$, $E^\bullet(\lambda)$ and $F^\bullet(\lambda)$ in terms of the blocks of $U(\lambda)$ that are given in Table 7.3. Then it is readily checked that:

$$E_{U_\#} = (G_U)^\#, \qquad F_{U_\#} = (H_U)^\#, \qquad E^\bullet_{U_\#} = (G^\bullet_U)^\#, \qquad F^\bullet_{U_\#} = (H^\bullet_U)^\# \quad \text{and} \quad z_{U_\#} = c_U. \tag{7.28}$$

It is now relatively easy to extract criteria for left regularity from the already established criteria for right regularity.

THEOREM 7.5. *Let $U \in \mathscr{U}(J)$ for J equal to j_p, J_p or $\mathscr{J}_p$. Then the following statements are equivalent*:

1. *$U(\lambda)$ is left strongly regular.*
2. *$E_U(\mu)\, E_U(\mu)^*$ satisfies the matricial Muckenhoupt condition* (4.2) *and $U(\mu) \in \widetilde{L_2^{m\times m}}(\Omega_0)$.*
3. *$E_U(\mu)\, E_U(\mu)^*$ satisfies the matricial Muckenhoupt condition* (4.2) *and the singular component $(z_U)_s(\lambda)$ of the Carathéodory mvf $z_U(\lambda)$ is constant.*

Now let $\psi_-(\lambda)$ and $\psi_+(\lambda)$ be solutions of the factorization problem

$$\psi_-(\mu)\,\psi_-(\mu)^* = \psi_+(\mu)\,\psi_+(\mu)^* = E_U(\mu)\, E_U(\mu)^* \tag{7.29}$$

such that

$$(\psi_-^\#)^{\pm 1} \in \mathscr{N}_+^{p\times p}(\Omega_+) \quad \text{and} \quad (\psi_+)^{\pm 1} \in \mathscr{N}_+^{p\times p}(\Omega_+), \tag{7.30}$$

and let

$$f_U(\mu) = \psi_+(\mu)^{-1}\, \psi_-(\mu). \tag{7.31}$$

THEOREM 7.6. *Let $U \in \mathscr{U}(J)$ for $J = j_p$, J_p or $\mathscr{J}_p$. Then the following statements are equivalent*:

1. *$U \in \mathscr{U}_{\ell R}(J)$.*
2. $\operatorname{index}\{f_U\} = 0$ *and $(z_U)_s(\lambda)$ is constant.*

The formula for $W(\lambda)$ alias $U(\lambda)$ in the first column of Table 7.3 can also be expressed in the form

$$W(\lambda)=\frac{1}{2}\begin{bmatrix} \{I_p+z^{\#}(\lambda)\}\,\psi_-(\lambda) & -\{I_p-z(\lambda)\}\,\psi_+(\lambda) \\ -\{I_p-z^{\#}(\lambda)\}\,\psi_-(\lambda) & \{I_p+z(\lambda)\}\,\psi_+(\lambda) \end{bmatrix} \times\begin{bmatrix} b_5(\lambda) & 0 \\ 0 & b_6(\lambda)^{-1} \end{bmatrix}, \tag{7.32}$$

where $b_5(\lambda)$ and $b_6(\lambda)$ are $p\times p$ inner mvf's that emerge from the inner-outer factorization of E^{-1} and $(F^{\#})^{-1}$ and formulas (7.19) and (7.29) with $E(\lambda)=E_U(\lambda)$:

$$E(\lambda)^{-1}=b_6(\lambda)\,\psi_+(\lambda)^{-1} \quad \text{and} \quad (F^{\#})(\lambda)^{-1}=\psi_-^{\#}(\lambda)^{-1}\,b_5(\lambda). \tag{7.33}$$

These factorizations exist because

$$(\rho_\omega E)^{-1}\in H_2^{p\times p}(\Omega_+) \qquad \text{and} \qquad (\rho_\omega F^{\#})^{-1}\in H_2^{p\times p}(\Omega_+) \tag{7.34}$$

for every $\omega\in\Omega_+$.

The parametrization formula (7.32) is dual to the parametrization formula (6.30). It can also be obtained by playing with the mapping $W\to W_{\#}$. Upon traversing this second route, the inner mvf's $b_5(\lambda)$ and $b_6(\lambda)$ are obtained from the factorization of the block diagonal entries $s_{11}(\lambda)$ and $s_{22}(\lambda)$ of the Potapov–Ginzburg transform of $W(\lambda)$:

$$s_{11}(\lambda)=\varphi_5(\lambda)\,b_5(\lambda) \qquad \text{and} \qquad s_{22}(\lambda)=b_6(\lambda)\,\varphi_6(\lambda), \tag{7.35}$$

where $\varphi_j\in\mathcal{S}_{out}^{p\times p}$ and $b_j\in\mathcal{S}_{in}^{p\times p}$ for $j=5,\ 6$.

Finally, upon invoking formula (7.26), we see that

$$\begin{aligned} T_{W_\#}[I_p] &= (w_{11}-w_{12})^{-1}\,(-w_{21}+w_{11}) \\ &= b_6\psi_+^{-1}\psi_-b_5 \\ &= b_6 f_U b_5 \end{aligned}$$

must belong to $\mathcal{S}_{in}^{p\times p}$.

Thus, an arbitrary mvf $W\in\mathcal{U}(j_p)$ may be parametrized by any $p\times p$ mvf $z(\lambda)$ that meets the conditions

$$z\in\Pi\cap\mathcal{C}^{p\times p}(\Omega_+) \qquad \text{and} \qquad z(\mu)+z(\mu)^*>0 \qquad \text{a.e. on } \Omega_+ \tag{7.36}$$

and any pair of $p\times p$ inner mvf's $b_5(\lambda)$ and $b_6(\lambda)$ such that

$$b_6(\lambda)\,\psi_+(\lambda)^{-1}\,\psi_-(\lambda)\,b_5(\lambda)\in\mathcal{S}_{in}^{p\times p}(\Omega_+), \tag{7.37}$$

where $\psi_-(\lambda)$ and $\psi_+(\lambda)$ are solutions of the factorization problem

$$\psi_-(\mu)\,\psi_-(\mu)^* = \psi_+(\mu)\,\psi_+(\mu)^* = 2\{z(\mu) + z(\mu)^*\}^{-1} \tag{7.38}$$

a.e. on Ω_0 that satisfy the conditions specified in (7.30). This parametrization arises in the investigation of the Darlington realization that was mentioned earlier in Section 6.

The condition (7.37) states that the ratio of the mvf's $E(\lambda)$ and $F(\lambda)$ defined in Table 7.3 is inner:

$$E(\lambda)^{-1}\,F(\lambda) \in \mathcal{S}_{in}^{p\times p}(\Omega_+). \tag{7.39}$$

Thus, the pair $\{E, F\}$ is characterized by the properties (7.34), (7.39) and the fact that

$$E \in \Pi^{p\times p} \qquad \text{and} \qquad F \in \Pi^{p\times p}. \tag{7.40}$$

An analogous characterization can be given for the pair $\{G, H\}$, either directly from the definitions or via (7.28). Similar remarks apply to the pairs $\{E^\bullet, F^\bullet\}$ and $\{G^\bullet, H^\bullet\}$.

Remark 7.7. The duality principle can be considered for mvf's $W \in \mathcal{U}(j_{pq})$ with $q \neq p$ too. In this instance, we define

$$W_\#(\lambda) = \mathcal{J}_{pq}^*\, W^\#(\lambda)\, \mathcal{J}_{pq},$$

where

$$\mathcal{J}_{pq} = \begin{bmatrix} 0 & iI_p \\ -iI_q & 0 \end{bmatrix}$$

and is no longer a signature matrix when $q \neq p$. Then

$$W \in \mathcal{U}(j_{pq}) \Leftrightarrow W_\# \in \mathcal{U}(j_{qp}).$$

The characterization (7.26) of left strong regularity is valid for the setting $q \neq p$ also.

7.3. *A short detour on left γ-generating mvf's.* The existence of a right and left parametric representation $W \in \mathcal{U}(j_{pq})$: (6.30) and (7.32), suggest that it is reasonable to develop a dual theory of γ-generating mvf's. A four-block mvf

$$\mathfrak{B}(\mu) = \begin{bmatrix} \mathfrak{c}_-(\mu) & \mathfrak{d}_+(\mu) \\ \mathfrak{d}_-(\mu) & \mathfrak{c}_+(\mu) \end{bmatrix} \tag{7.41}$$

with diagonal blocks $\mathfrak{c}_-(\mu)$ of size $p \times p$ and $\mathfrak{c}_+(\mu)$ of size $q \times q$ is said to be a left γ-generating mvf if it has the following properties:

1. $\mathfrak{B}(\mu)$ is a measurable $m \times m$ mvf on Ω_0 and is j_{pq}-unitary a.e. on Ω_0.
2. $\mathfrak{c}_+(\mu)$ and $\mathfrak{c}_-(\mu)^*$ are the boundary values of mvf's $\mathfrak{c}_+(\lambda)$ and $\mathfrak{c}_-^{\#}(\lambda)$ that are holomorphic in Ω_+ and, in addition

$$\mathfrak{c}_+^{-1} \in \mathcal{S}_{out}^{q \times q}(\Omega_+) \qquad \text{and} \qquad (\mathfrak{c}_-^{\#})^{-1} \in \mathcal{S}_{out}^{p \times p}(\Omega_+). \tag{7.42}$$

3. The mvf

$$s(\mu) = \mathfrak{d}_+(\mu)\, \mathfrak{c}_+(\mu)^{-1} = [\mathfrak{c}_-(\mu)^*]^{-1}\, \mathfrak{d}_-(\mu)^* \tag{7.43}$$

is the boundary value of a mvf $s(\lambda) \in \mathcal{S}^{p \times q}(\Omega_+)$.

The class of such mvf's will be denoted $\mathfrak{M}_\ell(p, q)$. For each $\mathfrak{B} \in \mathfrak{M}_\ell(p, q)$, the (left) linear fractional transformation

$$T_{\mathfrak{B}}^{\ell}[\mathscr{E}] = (\mathfrak{c}_+ + \mathscr{E}\mathfrak{d}_+)^{-1}\, (\mathfrak{d}_- + \mathscr{E}\mathfrak{c}_-) \tag{7.44}$$

is well defined for $\mathscr{E} \in \mathcal{S}^{q \times p}(\Omega_+)$ and

$$\| T_{\mathscr{B}}^{\ell}[\mathscr{E}] \|_\infty \leqslant 1. \tag{7.45}$$

Moreover, as

$$T_{\mathfrak{B}}^{\ell}[\mathscr{E}] - T_{\mathfrak{B}}^{\ell}[\mathscr{E}^\circ] \in H_\infty^{q \times p}(\Omega_+) \tag{7.46}$$

for every choice of $\mathscr{E}$, $\mathscr{E}^\circ \in \mathcal{S}^{q \times p}(\Omega_+)$, we see that these linear fractional transformations are also intimately connected with the Nehari problem: If $f^\circ = T_{\mathfrak{B}}^{\ell}[\mathscr{E}^\circ]$, then

$$T_{\mathfrak{B}}^{\ell}[\mathcal{S}^{q \times p}] \subset \mathcal{F}(f^\circ), \tag{7.47}$$

the set of solutions tot he Nehari problem based on f°. This leads to classes $\mathfrak{M}_{\ell R}(p, q)$ of left regular and $\mathfrak{M}_{\ell sR}(p, q)$ of left strongly regular γ-generating mvf's. Correspondingly, the classes $\mathfrak{M}(p, q)$, $\mathfrak{M}_R(p, q)$ and $\mathfrak{M}_{sR}(p, q)$ that were studied earlier in this paper should now be renamed $\mathfrak{M}_r(p, q)$, $\mathfrak{M}_{rR}(p, q)$ and $\mathfrak{M}_{rsR}(p, q)$, respectively. All the results that were obtained earlier for these classes have analogues in the new settings.

7.4. *Entire J inner mvf's.* Theorem 7.2 gives necessary and sufficient conditions on $\Delta(\mu)$ under which the class $\mathcal{U}_\Delta(J) \neq \emptyset$. In a number of applications in the setting $\Omega_+ = \mathbb{C}_+$ it is of interest to know when $\mathcal{E} \cap \mathcal{U}_\Delta(J) \neq \emptyset$. The next theorem shows that

$$\mathcal{E} \cap \mathcal{U}_\Delta(J) \neq \emptyset \Leftrightarrow \Delta(\lambda) \text{ meets the three conditions in (6.32)} \\ \text{and } \Delta(\lambda) \text{ is entire.} \tag{7.48}$$

In view of Krein's characterization of the class $\mathcal{E} \cap \Pi^{m\times m}$ [Kr] ([RoR] is a convenient reference), the equivalence (7.48) can be reformulated as follows:

THEOREM 7.8. *Let $\Omega_+ = \mathbb{C}_+$, let J be equal to one of the signature matrices j_p, J_p or $\mathcal{J}_p$. Then the class $\mathcal{E} \cap \mathcal{U}_\Delta(J) \neq \emptyset$ if and only if the mvf $\Delta(\lambda)$ meets the following five conditions*:

1. $\Delta \in \mathcal{E}^{p\times p}$.
2. $\Delta(\mu) > 0$ *on* $\mathbb{R}$.
3. $\Delta^{-1} \in \widetilde{L_1^{p\times p}}(\mathbb{R})$.
4. $\Delta(\lambda)$ *is of exponential type.*
5. $\Delta(\lambda)$ *is of Cartwright class, i.e.,* $\log|\det \Delta(\mu)| \in \widetilde{L_1}(\mathbb{R})$.

Proof. Suppose first that $\mathcal{E} \cap \mathcal{U}_\Delta(J) \neq \emptyset$ and let $\mathcal{U} \in \mathcal{E} \cap \mathcal{U}_\Delta(J)$. Then from Table 7.1 and formula (7.7), we see that

$$\Delta(\lambda) = G^\#(\lambda)\, G(\lambda) \tag{7.49}$$

is an entire mvf. By Theorem 7.2, $\Delta(\lambda)$ meets the three conditions in (6.32). Therefore (3) holds and $\Delta(\mu) \geqslant 0$ on $\mathbb{R}$. However, in view of (1) and (3), the inequality must be strict, i.e., (2) holds. Since every $U \in \mathcal{E} \cap \mathcal{U}(J)$ is of exponential type (see e.g., the estimates in [ArD1] and/or Krein's theorem [Kr]), it follows from Table 7.2 that $G(\lambda)$ and (hence) $\Delta(\lambda)$ are of exponential type, i.e., (4) holds. Moreover, since $\Delta \in \Pi^{p\times p}$, $\det \Delta \in \Pi$ and consequently (5) holds.

Conversely, let $\Delta(\lambda)$ be a $p\times p$ mvf that satisfies the stated five conditions and let $J = J_p$. Then, by Theorem 2.3 in [Ar3], the mvf $c_a(\lambda)$ admits the Darlington representation

$$c_a(\lambda) = T_A[I_p]$$

for some $A \in \mathcal{E} \cap \mathcal{U}(J_p)$, i.e., in terms of the block decomposition $A(\lambda) = [a_{ij}(\lambda)]$, $i, j = 1, 2$,

$$c_a(\lambda) = \{a_{11}(\lambda) + a_{12}(\lambda)\}\{a_{21}(\lambda) + a_{22}(\lambda)\}^{-1}.$$

Thus, if

$$U(\lambda) = j_p A^{\#}(\lambda)\, j_p, \tag{7.50}$$

we obtain exactly the representation for $c_a(\lambda)$ in terms of the blocks of $U(\lambda)$ that is given in the second row of Table 7.2 for $J = J_p$ and $c(\lambda) = c_a(\lambda)$. Therefore $U \in \mathscr{E} \cap \mathscr{U}_\Delta(J_p)$. The same conclusions clearly hold for the other two choices of J. ∎

Remark 7.9. The mvf $U(\lambda) = j_p A^{\#}(\lambda)\, j_p$ that was exhibited in the proof of the last theorem is in fact equal to $U_a(\lambda)$. But this in turn implies that

$$U_a(\lambda)\, U_s(\lambda) \in \mathscr{E} \cap \mathscr{U}_\Delta(J_p)$$

for every choice of $U_s(\lambda)$ in Table 7.2, providing that $c_s(\lambda)$ is restricted to the form

$$c_s(\lambda) = 2i\gamma + \lambda\beta$$

for some choice $\beta \in \mathbb{C}^{p\times p}$ with $\beta \geqslant 0$ and $\gamma \in \mathbb{C}^{p\times p}$ with $\gamma = \gamma^*$. This restriction on the form is necessary for $U_s(\lambda)$ to be entire.

Remark 7.10. We further remark that there is an analogue of Theorem 7.8 that can be formulated in terms of the mvf $\Delta^\bullet(\lambda)$ and also in terms of the weights $E(\lambda)\, E^{\#}(\lambda)$ and $E^\bullet(\lambda)(E^\bullet)^{\#}\,(\lambda)$ that play the role of $\Delta(\lambda)$ and $\Delta^\bullet(\lambda)$ in Table 7.3.

7.5. *Constant* $c_s(\lambda)$ *and* $z_s(\lambda)$. It is well known (and readily checked) that if $c \in \mathscr{C}^{p\times p}(\Omega_+)$, then

$$\begin{aligned} &(\Re c)(\omega) = 0 \qquad \text{for a point} \quad \omega \in \Omega_+ \\ \Leftrightarrow\ &(\Re c)(\omega) = 0 \qquad \text{for every point} \quad \omega \in \Omega_+. \end{aligned}$$

Thus, $c_s(\lambda)$ will be constant if and only if $(\Re c_s)(\omega) = 0$ for at least one point $\omega \in \Omega_+$. This translates to the following condition:

$c_s(\lambda)$ is constant in Ω_+ if and only if

$$\begin{aligned} &(\Re c)(0) = (\Re c_a)(0) = \frac{1}{2\pi}\int_0^{2\pi} \Delta(e^{i\theta})^{-1}\, d\theta \quad \text{if } \Omega_+ = \mathbb{D} \\ &(\Re c)(i) = (\Re c_a)(i) = \frac{1}{\pi}\int_{-\infty}^{\infty} (1+\mu^2)^{-1}\, \Delta(\mu)^{-1}\, d\mu \quad \text{if } \Omega_+ = \mathbb{C}_+. \end{aligned} \tag{7.51}$$

Similar conditions apply to $z_s(\lambda)$, with $E(\mu)\, E(\mu)^*$ in place of $\Delta(\mu)$.

If $\Omega_+ = \mathbb{C}_+$ and $U \in \mathcal{U}(J)$ is meromorphic in $\mathbb{C}$, then the mvf's $c(\lambda)$ and $z(\lambda)$ that are defined in Tables 7.1 and 7.3, respectively, are also meromorphic in $\mathbb{C}$. In this case

$$c_s(\lambda) = 2i\gamma - i\beta\lambda + \frac{1}{i}\sum_j \left\{\frac{1}{\mu_j - \lambda} - \frac{\mu_j}{1+\mu_j^2}\right\}\sigma_j,$$

where $\gamma = \gamma^*$, $\beta \geqslant 0$, $\sigma_j \geqslant 0$ and the summation is taken over the real poles μ_j of $c(\lambda) = H(\lambda)^{-1} H^{\bullet}(\lambda)$. There is a similar formula for $z_s(\lambda)$ in terms of the real poles of $z(\lambda) = E^{\bullet}(\lambda) E(\lambda)^{-1}$. Thus, $c_s(\lambda)$ [resp. $z_s(\lambda)$] will be constant if and only if

$$c(\lambda) \qquad [\text{resp. } z(\lambda)] \qquad \text{is holomorphic on } \mathbb{R} \tag{7.52}$$

and

$$\lim_{\nu \uparrow \infty} \frac{c(i\nu)}{\nu} = 0 \qquad \left[\text{resp. } \lim_{\nu \uparrow \infty} \frac{z(i\nu)}{\nu} = 0\right]. \tag{7.53}$$

If $U \in \mathcal{U}(J)$ is also an entire mvf, then the first constraint (7.52) is automatically met. Thus $c_s(\lambda)$ [resp. $z_s(\lambda)$] will be constant if and only if (7.53) holds.

Adapting the terminology of Golinskii and Mikhailova [GoMi] to the present setting, we shall say that a mvf $U \in \mathcal{U}(J)$ is a de Branges matrix if it is meromorphic in $\mathbb{C}$ and $z(\lambda)$ is holomorphic on $\mathbb{R}$. Furthermore, we shall say that a de Branges matrix is perfect if

$$\lim_{\nu \uparrow \infty} \frac{z(i\nu)}{\nu} = 0. \tag{7.54}$$

If $U \in \mathcal{E} \cap \mathcal{U}(J)$ and $J = J_p$ or $J = \mathcal{J}_p$, then (in the terminology of [DyI]) $\{E, F\}$ and $\{E^{\bullet}, F^{\bullet}\}$ are "de Branges pairs", i.e., they generate reproducing kernel Hilbert spaces of $p \times 1$ vector valued entire functions of the kind considered by the Branges in [dB3] for $p = 1$ and in [dB4] for $p > 1$. In particular this means that the inequality

$$E(\lambda)\, E(\lambda)^* - F(\lambda)\, F(\lambda)^* > 0 \tag{7.55}$$

holds for every point $\lambda \in \mathbb{C}_+$. If $J = \mathcal{J}_p$ and $\overline{U(\bar{\lambda})} = U(\lambda)$, then it follows from the formulas in Table 7.3 that

$$\overline{E(\bar{\lambda})} = F(\lambda). \tag{7.56}$$

Thus the inequality (7.55) can be reformulated as

$$E(\lambda)\, E(\lambda)^* > \overline{E(\bar{\lambda})}\, \overline{E(\bar{\lambda})^*}. \tag{7.57}$$

In the scalar case this reduces to the well known de Branges inequality

$$|E(\lambda)| > |E(\bar{\lambda})| \tag{7.58}$$

for $\lambda \in \mathbb{C}_+$.

7.6. *An example.* In this subsection we shall use the characterization of the class $\mathcal{U}_{sR}(J)$ that was established in Theorem 7.2 to show that in general, the inclusion

$$L_\infty^{m\times m}(\Omega_0) \cap \mathcal{U}(J) \subset \mathcal{U}_{sR}(J) \tag{7.59}$$

is proper. In particular, we shall exhibit a one-parameter family of entire strongly regular 2×2 mvf's that are J-inner with respect to $\mathbb{C}_+$ for which the corresponding weight $\Delta(\lambda)$ is a scalar entire function that is positive and unbounded on $\mathbb{R}$ yet satisfies the Muckenhoupt condition

$$\frac{1}{(b-a)^2}\int_a^b \Delta(\mu)\, d\mu \int_a^b \Delta(\mu)^{-1}\, d\mu \leqslant \kappa < \infty \tag{7.60}$$

for every $b > a$.

LEMMA 7.11. *Let $-1 < \alpha < 1$. Then the function*

$$\omega(\mu) = |\mu|^\alpha \tag{7.61}$$

satisfies the scalar Muckenhoupt condition. Moreover, if there exist a set of positive numbers $M_1, \ldots, M_4$ and c such that

$$0 < M_1 \leqslant \Delta(\mu) \leqslant M_2 \qquad \textit{for} \quad |\mu| \leqslant c \tag{7.62}$$

and

$$M_3\, |\mu|^\alpha \leqslant \Delta(\mu) \leqslant M_4\, |\mu|^\alpha \qquad \textit{for} \quad |\mu| > c, \tag{7.63}$$

then $\Delta(\mu)$ also satisfies the scalar Muckenhoupt condition (7.60).

Proof. The verification is by tedious but elementary estimates of the integral's in (7.60) for the different cases correspond to the position of the interval $[a, b]$ with respect to the point zero for the first statement and with respect to the interval $[-c, c]$ for the second statement. ▮

Let

$$S_t(\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n+1}}{\Gamma(1+t+2n)}. \tag{7.64}$$

This function has been investigated extensively by M. M. Dzhrbashyan [Dz] in his study of interpolation problems for entire functions of finite order and finite type that belong to L_2 with weight $\omega(\mu)$. In [Dz] it is shown, with the help of the integral representation formulas

$$S_t(\lambda) = \begin{cases} \dfrac{\lambda}{\Gamma(t)} \displaystyle\int_0^1 (1-x)^{t-1} \cos \lambda x \, dx & \text{for} \quad 0 < t \leqslant 1 \\ \dfrac{1}{\Gamma(t-1)} \displaystyle\int_0^1 (1-x)^{t-2} \sin \lambda x \, dx & \text{for} \quad 1 < t < 2, \end{cases} \tag{7.65}$$

that $S_t(\lambda)$ is an entire function of exponential type and that for $0 < t < 2$, $S_t(\lambda)$ has real simple roots and for $\mu > 0$,

$$S_t(\mu) = \mu^{1-t} \cos\left(\mu - \frac{\pi}{2} t\right) + O(\mu^{-1}) \qquad \text{as} \quad \mu \to +\infty. \tag{7.66}$$

Our next objective is to obtain analogous asymptotic formulas for

$$S_t'(\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, \lambda^{2n}}{\Gamma(1+t+2n)}, \tag{7.67}$$

the derivative of $S_t(\lambda)$ with respect to λ.

Lemma 7.12. *If $0 < t < 1$, then*

$$S_t'(\lambda) = (1-t) \frac{S_t(\lambda)}{\lambda} + \frac{1}{\Gamma(t)} - \lambda S_{t+1}(\lambda)$$

and, for $\mu > 0$,

$$S_t'(\mu) = \mu^{1-t} \sin\left(\mu - \frac{\pi}{2} t\right) + O(1) \qquad \text{as} \quad \mu \uparrow \infty.$$

Proof. If $0 < t \leqslant 1$, then, upon differentiating the corresponding integral representation formula for $S_t(\lambda)$ with respect to λ we obtain

$$S_t'(\lambda) = \frac{S_t(\lambda)}{\lambda} + ① + ②,$$

where

$$\begin{aligned}
① &= \frac{\lambda}{\Gamma(t)} \int_0^1 (1-x)^t \sin \lambda x \, dx \\
&= -\frac{1}{\Gamma(t)} \int_0^1 (1-x)^t \left\{ \frac{\partial}{\partial x} \cos \lambda x \right\} dx \\
&= \frac{1}{\Gamma(t)} - \frac{t}{\Gamma(t)} \int_0^1 (1-x)^{t-1} \cos \lambda x \, dx \\
&= \frac{1}{\Gamma(t)} - t \frac{S_t(\lambda)}{\lambda}
\end{aligned}$$

and

$$\begin{aligned}
② &= -\frac{\lambda}{\Gamma(t)} \int_0^1 (1-x)^{t-1} \sin \lambda x \, dx \\
&= -\frac{\lambda}{\Gamma(t)} \int_0^1 (1-x)^{t+1-2} \sin \lambda x \, dx \\
&= -\lambda S_{t+1}(\lambda).
\end{aligned}$$

The second statement now follows easily from formula (7.66). ▮

LEMMA 7.13. *If* $1 < t < 2$ *and* $\mu > 0$, *then*

$$\begin{aligned}
S_t'(\mu) &= \frac{S_{t-1}(\mu)}{\mu} + o(1) \qquad \textit{as} \quad \mu \uparrow \infty \\
&= -\mu^{1-t} \sin\left(\mu - \frac{\pi}{2} t\right) + o(1) \qquad \textit{as} \quad \mu \uparrow \infty.
\end{aligned}$$

Proof. If $1 < t < 2$, then, upon differentiating the corresponding integral representation formula with respect to λ, we obtain the formula

$$\begin{aligned}
S_t'(\lambda) &= \frac{1}{\Gamma(t-1)} \int_0^1 (1-x)^{t-2} x \cos \lambda x \, dx \\
&= -\frac{1}{\Gamma(t-1)} \int_0^1 (1-x)^{t-1} \cos \lambda x \, dx + \frac{S_{t-1}(\lambda)}{\lambda}.
\end{aligned}$$

The rest is clear from the Riemann–Lebesgue lemma. ▮

Let

$$f_t(\lambda) = S_t(\lambda) + i S_t'(\lambda). \tag{7.68}$$

Then

$$|f_t(\mu)|^2 = |S_t(\mu)|^2 + |S_t'(\mu)|^2 \tag{7.69}$$

for $\mu \in \mathbb{R}$ and

$$\lim_{|\mu| \uparrow \infty} \frac{|f_t(\mu)|^2}{|\mu|^{2-2t}} = 1 \qquad \text{for} \quad 0 < t < 2. \tag{7.70}$$

For $t \neq 1$, the last formula follows from Lemmas 7.12 and 7.13 and the fact that

$$S_t(-\lambda) = -S_t(\lambda) \qquad \text{and} \qquad S_t'(-\lambda) = S_t(\lambda).$$

However, the formula is selfevident for $t = 1$, since

$$f_1(\lambda) = ie^{-i\lambda}.$$

THEOREM 7.14. *Let*

$$\Delta_t(\lambda) = f_t(\lambda)\, f_t^{\#}(\lambda) \tag{7.71}$$

and let $\frac{1}{2} < t < \frac{3}{2}$. *Then*:

1. $\mathscr{E} \cap \mathscr{U}_{\Delta_t} \cap \mathscr{U}_{sR}(J) \neq \emptyset$.
2. *If* $U \in \mathscr{U}_{\Delta_t}$ *and* $t \neq 1$, *then* $U \notin L_\infty^{2\times 2}(\mathbb{R})$.

Proof. Clearly $\Delta_t(\lambda)$ is an entire function of exponential type. Since the roots of $S_t(\lambda)$ are real and simple, $\Delta_t(\mu) > 0$ on $\mathbb{R}$. Moreover, in view of (7.70) and Lemma 7.11, $\Delta_t(\mu)$ satisfies the Muckenhoupt condition (7.60), $\Delta_t^{-1} \in \widetilde{L}_1(\mathbb{R})$ and $\Delta_t(\lambda)$ is of Cartwright class. Therefore, by Theorem 7.8, the class $\mathscr{E} \cap \mathscr{U}_{\Delta_t}(J) \neq \emptyset$. Moreover, by Theorem 7.1, every $U \in \mathscr{E} \cap \mathscr{U}_{\Delta_t}(J)$ for which $c_s(\lambda)$ is constant, is strongly regular. This completes the proof of the first statement.

The second statement follows from the observation that if $U \in L_\infty^{2\times 2}(\mathbb{R})$, then $\Delta(\mu)^{\pm 1} \in L_\infty(\mathbb{R})$. However, this contradicts the asymptotic formula (7.70) when $t \neq 1$. ▮

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